Intrinsic irreversibility and the validity of the kinetic description of chaotic systems

Hiroshi H. Hasegawa and Dean J. Driebe

Center for Studies in Statistical Mechanics and Complex Systems, University of Texas at Austin, Austin, Texas 78712 and International Solvay Institutes for Physics and Chemistry, 1050 Brussels, Belgium

(Received 28 July 1993; revised manuscript received 19 April 1994)

Irreversibility for a class of chaotic systems is seen to be an exact consequence of the dynamics through the use of a generalized spectral representation of the time evolution operator of probability densities. The generalized representation is valid for one-dimensional systems when the initial probability density satisfies certain "physical conditions" of smoothness. The formalism is first applied to the one-dimensional multi-Bernoulli map, which is a simple map displaying deterministic diffusion. The two-dimensional, invertible baker and multibaker transformations are then studied and the physical conditions determining which discrete spectral values are realized are seen to depend on the smoothness of both the density as well as the observable considered. The generalized representation is constructed using a resolvent formalism. The eigenstates of the diffusive systems are seen to be of a fractal nature.

PACS number(s): 05.45.+b, 05.20.Dd, 05.70.Ln, 05.60.+w

I. INTRODUCTION

In the world around us irreversibility plays a fundamental role. On the other hand, the basic dynamical laws of physics are expressed as time-reversible equations, where time plays only a parametric role. For some years it has been recognized that unstable dynamical systems are the most typical in nature and that they play an essential role in the elucidation of irreversibility [1]. Recently, irreversibility for classes of chaotic systems has been discussed using the explicit construction of the generalized spectral resolution [2–4]. The main idea is that irreversibility can be understood from the intrinsic properties of the dynamics without using extradynamical arguments such as coarse-graining. For unstable systems the concept of a trajectory loses operational meaning, leading one to consider the evolution of probability densities. Irreversible behavior, such as approach to equilibrium, can be understood from the purely mathematical spectral properties [5,6] of the full time evolution operator for densities. For a class of systems the time evolution of probability densities is explicitly decomposed into a sum of exponentially decaying eigenmodes—the generalized spectral resolution [7].

Since the dynamics is time reversible, we need to specify a supplementary condition to decompose the time evolution into exponentially decaying eigenmodes. Ordinarily, a kinetic equation is derived using approximation schemes or extradynamical arguments such as coarse-graining. We assert that the supplementary conditions for irreversibility have to be determined from the dynamics. For certain chaotic systems Ruelle [5], Pollicott [6], Baladi and Keller [8], and Rugh [9] showed that under the condition of smoothness of observables, the spectral radius is bounded exponentially and only certain discrete spectra are realized. This means that physical time scales which characterize irreversibility have been made explicit by the introduction of this condition.

The decomposition of the time evolution of probability densities into a sum of exponentially decaying modes was introduced by the Brussels group for thermodynamic systems using a weak coupling expansion [10]. For maps the time evolution operator of densities is known as the Frobenius-Perron operator [11]. Mori, So, and Ose [12], Dörfle [13], and Roepstorff [14] studied exponentially decaying eigenstates of the Frobenius-Perron operator of some simple dissipative chaotic systems. Their eigenvalues are related to the Ruelle-Pollicott resonances [5,6]. These resonances are the zeros of the Ruelle ζ function and may be obtained by counting the periodic orbits. From this point of view, Dana [15] obtained the diffusion coefficient for some simple Hamiltonian systems and Christiansen, Paladin, and Rugh [16], Artuso [17], and Gaspard [18] obtained the generalized spectrum for some highly chaotic systems.

In this paper our main interest is in chaotic systems which display thermodynamic behavior such as diffusion. For the systems we consider, there exists a simple intertwining relation between the Frobenius-Perron operator and the derivative operator. Using the intertwining relation, we can show under which conditions the irreversible kinetic description becomes valid, even when the underlying trajectory dynamics may be time reversible. The condition is essentially m-times differentiability of the probability density for a one-dimensional system. For a two-dimensional system it is necessary to have differentiability for both the probability density and the observable. These conditions reduce the spectral radius as $\exp[-m\lambda]$, where λ is the Lyapunov exponent, and introduce physical time scales into the spectrum. In the limit of $m \to \infty$, we recover the result of the general argument by Rugh [9] for hyperbolic analytic maps.

As Artuso [17] and Gaspard [18] calculated for diffusive systems, the generalized spectrum can be obtained as zeros of the Ruelle ζ functions by counting periodic orbits. In this paper we explicitly construct the eigenstates using a method based on the Euler-Maclaurin expansion and the intertwining relation. Since the eigenstates give us the coefficients of the exponentially decaying modes, the explicit construction becomes important for comparison with experimental data.

The one-dimensional dyadic Bernoulli map forms the basis of our analysis so we review it in Sec. II. In the Bernoulli map the evolution of a probability density is just a simple decay to the equilibrium state of a constant density. "Physical" decaying eigenstates have been constructed which are associated with discrete eigenvalues corresponding to physically observed time scales. The degree of smoothness of the initial density determines which discrete decaying modes are realized [4,19]. A spectral decomposition of the density in terms of the determined physical eigenstates and a remaining contribution is seen to correspond to a Euler-Maclaurin expansion.

In Sec. III we consider a one-dimensional model of deterministic diffusion which is constructed by coupling a chain of Bernoulli maps. It is referred to as the multi-Bernoulli map [20–22]. The evolution of this map is closer to more realistic thermodynamic systems as the dominant mode is diffusive. The one-dimensional maps that we study are not governed by unitary evolutions since they are noninvertible. The adjoint of their Frobenius-Perron operator is isometric and so they share many mathematical features with the unitary case. The physical behavior of the one-dimensional maps is interesting in its own right also.

We construct physical eigenstates of the multi-Bernoulli map using a resolvent formalism and an expansion in terms of a basis of eigenstates of the Bernoulli map. We discuss how the spectrum is determined by the smoothness properties of the functions in the domain of the Frobenius-Perron operator. The intertwining relation between the Frobenius-Perron operator and the derivative operator with respect to the spatial coordinate plays a key role in this analysis. The evolution of the system follows an approach to local equilibrium inside each cell, and then a global approach to equilibrium through diffusion between cells. Since we have the exact dispersion relation, any higher order diffusion coefficient, such as the Burnett coefficient, may be obtained. The left eigenstates of the multi-Bernoulli map are seen to have a fractal nature.

The two-dimensional area-preserving baker transformation is then studied using a two-dimensional, Euler-Maclaurin expansion. Here smoothness of both the initial density and the final observable are needed to determine the discrete spectral values that will be realized. The physical eigenvalues of the baker transformation are degenerate so one has associated eigenspaces instead of eigenstates. A projective decomposition of the resolvent is used to isolate contributions from the poles. The explicit form of the time correlation function from the first three poles is given and the connection with subdynamics [2,10] is discussed. Compact forms for the time correlation function are given in terms of a set of self-similar functions.

Finally, we consider the two-dimensional multibaker map [18,22] using the Euler-Maclaurin expansion. The multibaker map is the two-dimensional, area-preserving version of the multi-Bernoulli map. As in the multi-Bernoulli map, we use a discrete Fourier transform to separate the evolution into independent components. The analysis of the transformed multibaker map is quite similar to that of the baker map, but the form of the off-diagonal part of the Frobenius-Perron operator allows for more transitions. The explicit form of the time correlation from the first two poles is given.

The classic reversibility paradox and recurrence paradox, raised against an irreversible description of time-reversible dynamical systems, are completely resolved in the physical spectral representation. The time-reversible unitary evolution of the baker and multibaker transformations becomes irreversible. Since the probability density is irreducible to trajectories, the recurrence of points in phase space does not conflict with the fact that densities do not recur. Also, we are able to see explicitly for the systems considered here how the irreversible kinetic description arises from the instability of the underlying time-reversible dynamics. We expect that features of our analysis are applicable to a wide class of systems, including true Hamiltonian systems.

II. EVOLUTION OF DENSITIES IN CHAOTIC SYSTEMS

We are interested in investigating the dynamics of chaotic maps from the point of view of nonequilibrium statistical mechanics. Instead of following an individual iterate of the map, which would correspond to calculating the trajectory of a Hamiltonian system, we study the evolution of a probability density of iterates evolving under the map [11]. Even though the evolution of iterates is deterministic because we have a rule for determining the iterates, the sensitive dependence on initial conditions makes following the trajectory impossible from any practical point of view [23]. In this sense determinism is only a mathematical property for chaotic systems but not a physical property. It is thus natural to consider statistical properties of the iterates. We will see that when the domain of the time evolution operator of probability densities is restricted, the mathematical representation of the time evolution matches the physically observed behavior. The precise restriction that is necessary will be given for the individual models we will study.

The time evolution of a probability density ρ for a map f is governed by the Frobenius-Perron operator U, which advances the density by a unit step as

$$\rho(x,t+1)=U\rho(x,t)\equiv \sum_{\tilde{x}:x=f(\tilde{x})} \frac{\rho(\tilde{x},t)}{|f'(\tilde{x})|} , \tag{2.1}$$

where the sum is over the inverse branches of the possibly many-to-one map f (and we have assumed that the Jacobian of the transformation is 1). For invertible maps, U is unitary in a Hilbert space setting.

The adjoint of the Frobenius-Perron operator is the Koopman operator [11], $U^\dagger$, which gives the evolution of an observable $A(x)$ as

$$U^\dagger A(x)\equiv A(f(x)) . \tag{2.2}$$

The solution of the equation $U\rho^{\text{inv}}(x)=\rho^{\text{inv}}(x)$, i.e., the eigenstate of U with eigenvalue 1, gives the invariant density of the map from which time averages of observables may be obtained for maps which are ergodic. We refer to the invariant density as the equilibrium state of the map. (In this paper we always consider the measure to be Lebesgue measure. In general there may exist invariant densities corresponding to singular measures.)

A. The Bernoulli map

We first consider the analysis of the dyadic Bernoulli map which is a transformation on the unit interval given by the rule

$$x_{n+1}=f(x_n)=2x_n \pmod 1 . \tag{2.3}$$

The Bernoulli map is chaotic with Lyapunov exponent of ln2. Since the Bernoulli map is noninvertible, its evolution is not time reversible. We review here the analysis of the Bernoulli map and consider in Sec. III the noninvertible multi-Bernoulli map because their mathematical analysis is simpler than and they are the one-dimensional projections of the invertible baker and multibaker maps that will be considered in Secs. IV and V.

The Frobenius-Perron operator $\overline{U}_B$ for the Bernoulli map is given as

$$\rho(x,t+1)=\overline{U}_B\rho(x,t) = \frac{1}{2}\left[\rho\left[\frac{x}{2},t\right]+\rho\left[\frac{x+1}{2},t\right]\right] . \tag{2.4}$$

It is clear that the uniform density is the invariant density of $\overline{U}_B$. The evolution of a nonequilibrium density can be obtained from the spectral decomposition of $\overline{U}_B$.

The Koopman operator, $\overline{U}_B^\dagger$, for the Bernoulli map acts on an observable as

$$\overline{U}_B^\dagger A(x)=\begin{cases} A(2x) & \text{if } 0\le x<\frac{1}{2} \\ A(2x-1) & \text{if } \frac{1}{2}\le x<1 . \end{cases} \tag{2.5}$$

In the following we employ a Dirac bra-ket notation where $\langle f|g\rangle$ denotes the inner product of f and g:

$$\langle f|g\rangle \equiv \int_0^1 dx\, f^*(x)g(x) . \tag{2.6}$$

For an operator O we will write $\langle f|Og\rangle$ as $\langle f|O|g\rangle$ in the sense of its matrix elements. We will also formally write an operator O as $\sum_i |a_i\rangle\langle b_i|$ if matrix elements of O are written as

$$\langle f|O|g\rangle=\sum_i \langle f|a_i\rangle\langle b_i|g\rangle .$$

B. Spectral representations of the Bernoulli map

The spectral representations of the time evolution operator of probability densities evolving under the Bernoulli map have recently been studied by several authors [2–4]. Various spectral representations of the Frobenius-Perron operator are obtained depending on the functional space one is considering the operator to act in.

In the Hilbert space L_2 (on the unit interval) consider the states $e_{n,l}(x)$ defined by

$$e_{n,l}(x)=\exp[2\pi i 2^n(2l+1)x] , \tag{2.7}$$

where n is a non-negative integer and l is an integer [4,13,24]. Since any nonzero integer k can be written uniquely as $k=2^n(2l+1)$ for integers $n\ge 0$ and $-\infty<l<\infty$, $e_{n,l}(x)$ and 1 are the Fourier basis of $L_2(0,1)$. The states $e_{n,l}(x)$ are "shift states" of $\overline{U}_B$ as

$$\overline{U}_B e_{n,l}(x)=\begin{cases} e_{n-1,l}(x) & \text{if } n>0 \\ 0 & \text{if } n=0 . \end{cases} \tag{2.8}$$

From the shift states we can obtain the "coherent" eigenstate $\phi_{z,l}(x)$ of $\overline{U}_B$, with complex eigenvalue z [4,25], as

$$\phi_{z,l}(x)=\sum_{n=0}^{\infty} z^n e_{n,l}(x) . \tag{2.9}$$

For $|z|<1$, the series converges absolutely and uniformly and defines a continuous square-integrable function. The spectrum of $\overline{U}_B$ in L_2 being a closed set [26] therefore fills the unit disk $|z|\le 1$.

The Koopman operator, $\overline{U}_B^\dagger$ acts on the states (2.7) as the shift

$$\overline{U}_B^\dagger e_{n,l}(x)=e_{n+1,l}(x), \quad n\ge 0 \tag{2.10}$$

for which no eigenstates in L_2 (except for the trivial constant eigenstate with eigenvalue 1) can be constructed [27]. Since $\overline{U}_B^\dagger$ is isometric, i.e., $\overline{U}_B\overline{U}_B^\dagger=1$, it necessarily follows that it can only have eigenvalues of magnitude 1 in Hilbert space.

From the coherent states (2.9) a density can be constructed that decays at any rate [4,19]. For chaotic systems with a uniform stretching factor, one would expect the Lyapunov time, i.e., the inverse of the Lyapunov exponent, to play a role in the approach to the equilibrium density. In experimental observations (i.e., the power spectrum from a computer simulation) these are the physical time scales which are observed. As we will show below, for a smooth initial probability density the decay rates are uniquely determined and are characterized by the Lyapunov time. The decay rates correspond to poles in the Fourier transform of the correlation function and are naturally interpreted as resonances of the dynamical system [28]. The poles have been obtained for certain systems by Christiansen, Paladin, and Rugh [16], Artuso [17], and Gaspard [18], using periodic orbit theory and dynamical ζ functions. They are called "Ruelle-Pollicott resonances" [5,6].

Consider $\overline{U}_B^t\rho(x)$ expanded in terms of the shift states as

$$\overline{U}_B^t\rho(x)=\langle 1|\rho\rangle+\sum_{n=0}^{\infty}\sum_{l=-\infty}^{\infty} e_{n,l}(x)\langle e_{n+t,l}|\rho\rangle , \tag{2.11}$$

where we assumed $\rho(x)\in L_2$ and used the fact that 1 is the invariant density.

If $d\rho/dx\in L_2$, then, integrating by parts, we may rewrite

$$\langle e_{n+t,l}|\rho\rangle=\frac{1}{2^t}\frac{-1}{i2\pi 2^n(2l+1)}\times(\rho(1)-\rho(0)-\langle e_{n+t,l}|\frac{d}{dx}\rho\rangle)\ . \quad (2.12)$$

Using this equation, (2.11) is

$$\overline{U}_B^t\rho(x)=\langle 1|\rho\rangle+\frac{1}{2^t}\gamma^{(1)}(x)\int_0^1 dx'\frac{d}{dx'}\rho(x)-\frac{1}{2^t}\int_0^1 dx'\gamma^{(1)}(x-2^t x')\frac{d}{dx'}\rho(x')\ , \quad (2.13)$$

where we defined

$$\gamma^{(m)}(x)\equiv\sum_{k\neq 0}\frac{-1}{(2\pi ik)^m}e^{2\pi ikx} \quad (2.14)$$

for $m\geq 1$ and $\gamma^{(0)}(x)\equiv 1$.

If $d^M\rho/dx^M\in L_2$, then, by repeated integration by parts, we obtain

$$\overline{U}_B^t\rho(x)=\sum_{m=0}^{M}e^{-\gamma^{(m)}t}\gamma^{(m)}(x)\langle\tilde{\gamma}^{(m)}|\rho\rangle+e^{-\gamma^{(M)}t}\overline{\mathcal{R}}^{(M)}(x,t)\ , \quad (2.15)$$

where the exponents $\gamma^{(m)}=m\ln 2$, formally $\tilde{\gamma}^{(m)}(x)=d^m/dx^m$, and

$$\overline{\mathcal{R}}^{(M)}(x,t)\equiv-\int_0^1 dx'\gamma^{(M)}(x-e^{\gamma^{(M)}t}x')\frac{d^M}{dx'^M}\rho(x')\ . \quad (2.16)$$

Since

$$\|\overline{\mathcal{R}}^{(M)}(x,t)\|\leq\|d^M\rho/dx^M\|\ ,$$

the last term of the right-hand side in (2.15) is decaying as $(\frac{1}{2})^{Mt}$ or quicker. As a result, if $d^M\rho/dx^M\in L_2$, the slowest M exponentially decaying modes $(m=0,1,\ldots,M-1)$ are uniquely determined and their decay rates are $m\ln 2$. These correspond to the Ruelle-Pollicott resonances [2,5,6].

The expression (2.15) is a generalized spectral resolution of $\overline{U}_B$. The right eigenstate $\gamma^{(m)}(x)$ is equivalent to the mth Bernoulli polynomial $\beta_m(x)$ defined by the generating function [29]

$$ue^{xu}/(e^u-1)=\sum_{m=0}^{\infty}\beta_m(x)u^m\ .$$

We note that $\beta_m(x)$ are Bernoulli polynomials with a nonstandard normalization and the argument is taken modulo 1.

The left eigenstate $\tilde{\gamma}^{(m)}(x)$ can be interpreted as a generalized function on the space of smooth test functions expandable in terms of the right eigenstates. In the sense of Mikusiński and Boehme [30], the derivative operator is considered as a generalized function. More explicitly, $\tilde{\gamma}^{(m)}(x)$ is written in terms of the $(m-1)$-times derivative of the Dirac δ function as

$$\tilde{\gamma}^{(m)}(x)=(-1)^{m-1}[\delta^{(m-1)}(x-1+\epsilon)-\delta^{(m-1)}(x-\epsilon)]\ , \quad (2.17)$$

where ϵ is an infinitesimal positive number.

The mathematical structure of the generalized representation may be understood in the sense of a Gelfand triplet (rigged Hilbert space) [31,32], $\phi\subset\mathcal{H}\subset\phi^\dagger$. The left eigenstates $\tilde{\gamma}^{(m)}(x)$ belong to the larger functional space of Schwartz distributions $\phi^\dagger$, rather than to the ordinary Hilbert (L_2) space $\mathcal{H}$. Therefore, the initial probability density $\rho(x)$ should belong to the smooth test function space ϕ; otherwise, the generalized spectral resolution (2.15) is not valid. Clearly, a "point distribution," $\delta(x-x_0)$, it not applicable for (2.15). This is consistent with the fact that a trajectory shows no approach to equilibrium.

For initial densities which are entire functions of exponential type less than 2π, the value of M in (2.15) may be taken arbitrarily large. Taking $M\to\infty$, we obtain the complete decomposition into independent decaying modes [3]. This restriction on the density seems to us rather stringent from a physical point of view and so it is much more natural to consider the evolution as in (2.15). Also, the slowest decaying modes, which are dominant except for very short times, are determined when the density is just differentiable to a few orders.

C. The Euler-Maclaurin expansion

The expression (2.15) for $t=0$ corresponds to a Euler-Maclaurin expansion [3,33] of $\rho(x)$. For convenience we can rewrite (2.15) for $t=0$ as

$$\rho(x)=\sum_{m=0}^{M}\beta_m(x)\int_0^1 dx'\frac{d^m}{dx'^m}\rho(x')-\mathcal{B}_M(x)\cdot\int_0^1 dx'\mathbf{e}^*(x')\frac{d^M}{dx'^M}\rho(x')\ , \quad (2.18)$$

where

$$\mathcal{B}_M(x)\cdot\mathbf{e}(x')=\sum_k\beta_{M,k}(x)e_k(x')$$

and

$$\beta_{M,k}(x)=-e^{2\pi ikx}/(2\pi ik)^M$$

for $k\neq 0$, $\beta_{M,0}(x)=0$, and $e_k(x)=e^{2\pi ikx}$. Using a bra-ket notation, we can write the Euler-Maclaurin expansion of $f(x)$ up to Mth order as

$$f(x)=\sum_{m=0}^{M}\beta_m(x)\langle\tilde{\beta}_m|f\rangle-\mathcal{B}_M(x)\langle\mathbf{e}\tilde{\beta}_M|f\rangle\ , \quad (2.19)$$

where $\tilde{\beta}_m(x)\equiv d^m/dx^m=\tilde{\gamma}^{(m)}(x)$. In the space of functions whose Mth derivative belongs to L_2,

$$\sum_{m=0}^{M}|\beta_m\rangle\langle\tilde{\beta}_m|-|\mathcal{B}_M\rangle\langle\mathbf{e}\tilde{\beta}_M|=I_M \quad (2.20)$$

is a unit operator. We refer to the basis of this expansion as the Bernoulli basis (with remainder).

The Bernoulli map satisfies the intertwining relation between the Frobenius-Perron operator and the derivative operator of

$$(d/dx)\overline{U}_Bf(x)=\tfrac{1}{2}\overline{U}_B(d/dx)f(x)\ .$$

We may iterate this relation to obtain

$$\frac{d^m}{dx^m}\bar{U}_B^n f(x)=(\tfrac{1}{2})^{mn}\bar{U}_B^n\frac{d^m}{dx^m}f(x) \tag{2.21}$$

if $f(x)$ is at least m-times differentiable.

Using the Bernoulli basis, the intertwining relation, and that $\bar{U}_B^\dagger=1$, we reproduce (2.15) as

$$\begin{aligned}\rho(x,t)&=\bar{U}_B^t\rho(x,0)\\&=\sum_{m=0}^{M}\beta_m(x)\langle\tilde{\beta}_m|\bar{U}_B^t|\rho\rangle-\mathcal{B}_M(x)\langle e\tilde{\beta}_M|\bar{U}_B^t|\rho\rangle\\&=\sum_{m=0}^{M}e^{-\gamma^{(m)}t}\beta_m(x)\langle\tilde{\beta}_m|\rho\rangle\\&\quad-e^{-\gamma^{(M)}t}\mathcal{B}_M(x)\langle e\tilde{\beta}_0|\bar{U}_B^t\frac{d^M}{dx^M}|\rho\rangle\ .\end{aligned} \tag{2.22}$$

The intertwining relation (2.21) plays the crucial role in allowing us to use the Euler-Maclaurin expansion to obtain the independent decaying modes and a background term dependent on the smoothness of the density. The corresponding relation for the maps to be studied will also play a fundamental role in our analysis.

III. THE MULTI-BERNOULLI MAP

We have seen in Sec. II that the evolution of a density under the one-cell Bernoulli map is a simple exponential decay of modes, with constant decay rates, to the equilibrium state of a constant density. The irreversible behavior of more realistic thermodynamic systems is characterized by transport properties and hydrodynamic modes. In order to understand the dynamical origin of such behavior, we consider in this section and in Sec. V a system of chaotic maps which are coupled in order to allow transport of iterates throughout the composite system. These systems are models of deterministic diffusion.

We will see that the diffusive modes are modes of the Frobenius-Perron operator itself and that the diffusion coefficients are obtained from the eigenvalues of the Frobenius-Perron operator. Eigenstates associated with the other Ruelle-Pollicott resonances of the system are also constructed. The latter eigenstates decay with rates that are multiples of the Lyapunov exponent and so are quickly damped and represent a correction to diffusion for short times. Also, since we have the exact dispersion relation, the higher order diffusion coefficients, such as the Burnett coefficient, are also obtained.

The map that we study is made up of cells of Bernoulli maps that are coupled. It is referred to as the multi-Bernoulli map [20]. The map may be obtained in various ways, and we will see in Sec. V that it is the one-dimensional projection of the area-preserving, two-dimensional multibaker map. It belongs to the class of piecewise-linear periodic maps displaying deterministic diffusion [34]. Since the multi-Bernoulli map is not invertible, its evolution is necessarily irreversible, but the multibaker map is invertible and much of the analysis used here will be applicable to it. Some of the results of this section have been presented in two previous papers [20,21] of the authors.

The multi-Bernoulli map is given by the rule

$$X_{n+1}=\phi(X_n)=\begin{cases}4X_n-3q-1 & \text{if } q\le X_n<q+\tfrac{1}{2}\\ 4X_n-3q-2 & \text{if } q+\tfrac{1}{2}\le X_n<q+1\ ,\end{cases} \tag{3.1}$$

where q is an integer. Since the stretching factor is uniformly 4, the Lyapunov exponent of the map (3.1) is ln4. The multi-Bernoulli map is illustrated in Fig. 1.

We decompose X_n into its integer part q_n and its fractional part $x_n\in[0,1)$. The Frobenius-Perron operator $\bar{U}_{mB}$ of the multi-Bernoulli map (3.1) acts on a probability density as

$$\begin{aligned}\bar{U}_{mB}\rho(q+x,t)=\frac{1}{4}\Bigg[\rho\left[q-1+\frac{x}{4}+\frac{3}{4},t\right]&\\+\rho\left[q+\frac{x}{4}+\frac{1}{4},t\right]&\\+\rho\left[q+\frac{x}{4}+\frac{1}{2},t\right]&\\+\rho\left[q+1+\frac{x}{4},t\right]\Bigg]&\ .\end{aligned} \tag{3.2}$$

Considering the system on the interval $[0,L)$ with periodic boundary conditions, we separate the dynamics among unit intervals (i.e., in the variable q) from the internal motion (in x) in the intervals through the discrete Fourier transform pair

$$\rho(q+x,t)=\frac{1}{\sqrt{L}}\sum_{s=0}^{L-1}e^{i(2\pi/L)qs}\rho_s(x,t)\ , \tag{3.3a}$$

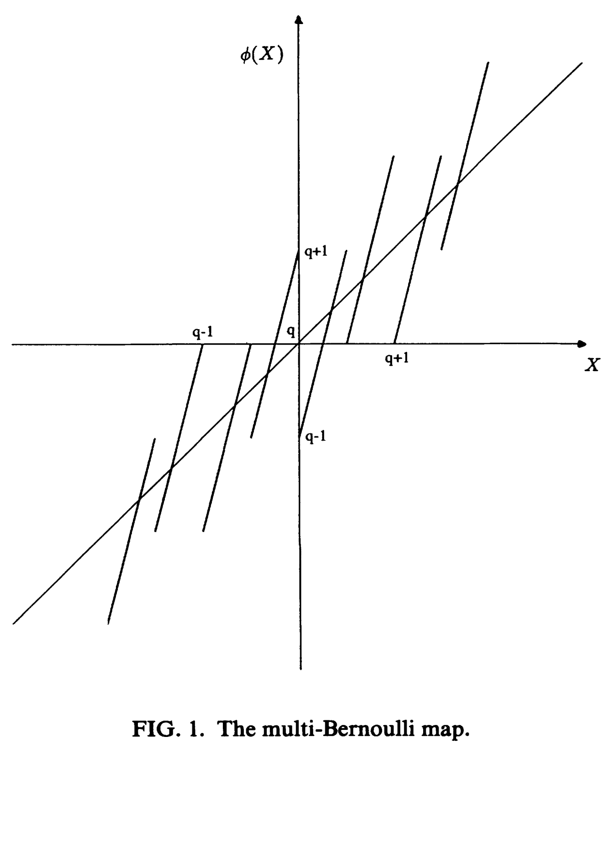

FIG. 1. The multi-Bernoulli map.

$$\rho_s(x,t)=\frac{1}{\sqrt{L}}\sum_{q=0}^{L-1} e^{-i(2\pi/L)qs}\rho(q+x,t) . \quad (3.3b)$$

Each mode ρ_s evolves independently as

$$\rho_s(x,t+1)=\frac{1}{4}\left[e^{i(2\pi/L)s}\rho_s\left[\frac{x}{4},t\right]+\rho_s\left[\frac{x+1}{4},t\right]+\rho_s\left[\frac{x+2}{4},t\right]+e^{-i(2\pi/L)s}\rho_s\left[\frac{x+3}{4},t\right]\right] . \quad (3.4)$$

This transformation corresponds to two successive applications of the operator $\overline{U}_s$, i.e.,

$$\rho_s(x,t+1)=\overline{U}_s^2\rho_s(x,t) ,$$

where $\overline{U}_s$ is defined by

$$\overline{U}_s\rho_s(x,t)=\frac{1}{2}\left[e^{i(\pi/L)s}\rho_s\left[\frac{x}{2},t\right]+e^{-i(\pi/L)s}\rho_s\left[\frac{x+1}{2},t\right]\right] . \quad (3.5)$$

Note that taking $s=0$ in (3.5) recovers the case of the dyadic Bernoulli map on the unit interval. We have thus decomposed the evolution of the density under the multi-Bernoulli map as

$$\rho(q+x,t)=\overline{U}_{mB}^t\rho(q+x,0)=\frac{1}{\sqrt{L}}\sum_{s=0}^{L-1} e^{i(2\pi/L)qs}\overline{U}_s^{2t}\rho_s(x,0) , \quad (3.6)$$

so that we may solve for the time evolution by considering the eigenvalue problem for $\overline{U}_s$.

Before turning to the construction of the generalized eigenstates, we briefly note that construction of formal coherent eigenstates of $\overline{U}_s$. The action of $\overline{U}_s$ on the states

$$e_{n,l}^{\sigma}(x)=\exp\{i2^n[2\sigma+2\pi(2l+1)]x\} , \quad (3.7)$$

where $\sigma=2\pi s/L$ and n and l are as in (2.7), is that of a *weighted* shift [27], i.e.,

$$\overline{U}_s e_{n,l}^{\sigma}(x)=\begin{cases} c_{n-1}^{\sigma}e_{n-1,l}^{\sigma}(x) & \text{if } n>0 \\ 0 \ \text{ if } n=0 , \end{cases} \quad (3.8)$$

where the weight factor

$$c_n^{\sigma}=e^{i2^n\sigma}\cos[\sigma(2^n-\tfrac{1}{2})] .$$

From the weighted shift states, we may thus construct the coherent eigenstate, $\phi_{z,l}^{\sigma}(x)$ of $\overline{U}_s$ with eigenvalue z as

$$\phi_{z,l}^{\sigma}(x)=\sum_{n=0}^{\infty}\frac{z^n}{\prod_{i=0}^{n-1}c_i^{\sigma}}e_{n,l}^{\sigma}(x) , \quad (3.9)$$

where the product in the denominator is one for $n=0$ and we assume that σ is such that c_n^{σ} is nonvanishing for all n.

As for the Bernoulli map discussed in Sec. II, in order to obtain a spectral representation of $\overline{U}_s$ that contains only the physically realizable eigenstates and eigenvalues, we must go to a generalized functional space. We now turn to the construction of the generalized spectral representation of $\overline{U}_s$.

By (3.5) we see that the transformed map is, except for the phase factors, just the dyadic Bernoulli map whose eigenstates are the Bernoulli polynomials and their duals. This suggests that a good "unperturbed" basis to choose is the Bernoulli basis. (In Ref. [20] the physical eigenstates of the multi-Bernoulli map were calculated using a basis of modified-Legendre polynomials.) The Frobenius-Perron operator (3.5) of the transformed map, $\overline{U}_s$, can be rewritten as

$$\overline{U}_s=\overline{U}_B e^{i(\pi s/L)r_1(x)}=e^{-\gamma_s^{(0)}}\overline{U}_B\left[1+i\tan\left[\frac{\pi s}{L}\right]r_1(x)\right] , \quad (3.10)$$

where $r_1(x)$ is the first Rademacher function $r_1(x)\equiv 1$ for $0\le x<\frac{1}{2}$ and $r_1(x)\equiv -1$ for $\frac{1}{2}\le x<1$, and, as will be shown later,

$$e^{-\gamma_s^{(m)}}\equiv\cos(\pi s/L)/2^m$$

are the eigenvalues of the exponentially decaying eigenstates of $\overline{U}_s$. The operator $\overline{U}_s$ is explicitly separated in (3.10) into parts that are diagonal and off-diagonal (as will be seen below) with respect to a basis of Bernoulli polynomials and their duals as

$$\overline{U}_s=\overline{U}_{s0}+\delta\overline{U}_s , \quad (3.11a)$$

where

$$\overline{U}_{s0}\equiv e^{-\gamma_s^{(0)}}\overline{U}_B , \quad (3.11b)$$

$$\delta\overline{U}_s\equiv i\tan\left[\frac{\pi s}{L}\right]e^{-\gamma_s^{(0)}}\overline{U}_B r_1(x)=i\sin\left[\frac{\pi s}{L}\right]\overline{U}_B r_1(x) . \quad (3.11c)$$

The matrix elements of $\overline{U}_s$ with respect to the Bernoulli basis (and the remainder part) are given in Appendix A. It is seen there that they are upper triangular so that transitions occur in only one direction and once a state is left it cannot be returned to.

The operator $\overline{U}_s$ satisfies, if $f(x)$ is differentiable, the intertwining relation

$$(d/dx)\overline{U}_s f(x)=\tfrac{1}{2}\overline{U}_s(d/dx)f(x) .$$

We will make repeated use of this relation in its iterated form of

$$\frac{d^m}{dx^m}\overline{U}_s^n f(x)=(\tfrac{1}{2})^{mn}\overline{U}_s^n\frac{d^m}{dx^m}f(x) , \quad (3.12)$$

if $f(x)$ is at least m-times differentiable.

A. Time evolution and spectral determination

The time evolution of the transformed probability density for the multi-Bernoulli map is given in the resolvent formalism by [2]

$$\rho_s(x,t)=\overline{U}_s^{2t}\rho_s(x,0)$$
$$=\frac{1}{2\pi i}\oint_{|z|=1+\epsilon} dz\, z^{2t}\frac{1}{z-\overline{U}_s}\rho_s(x,0)\ , \qquad (3.13)$$

where ϵ is an infinitesimal positive number so that the contour is taken just outside the unit circle. Eigenstates are obtained from (3.13) by enclosing poles of the resolvent analytically continued into the unit disk.

We now proceed to rewrite the resolvent acting on an initial density in terms of separate parts that are singular and regular in a region of the unit disk depending on the smoothness of the initial density. Inserting the Euler-Maclaurin expansion (2.20) on both sides of the resolvent gives

$$\frac{1}{z-\overline{U}_s}\rho_s(x,0)=\sum_{j'\ge j=0}^{M}\beta_j(x)\langle\tilde{\beta}_j|\frac{1}{z-\overline{U}_s}|\beta_{j'}\rangle\langle\tilde{\beta}_{j'}|\rho_s\rangle-\sum_{j=0}^{M}\beta_j(x)\langle\tilde{\beta}_j|\frac{1}{z-\overline{U}_s}|\mathcal{B}_M\rangle\langle \mathrm{e}\tilde{\beta}_M|\rho_s\rangle$$
$$+\mathcal{B}_M(x)\langle \mathrm{e}\tilde{\beta}_M|\frac{1}{z-\overline{U}_s}|\mathcal{B}_M\rangle\langle \mathrm{e}\tilde{\beta}_M|\rho_s\rangle\ , \qquad (3.14)$$

where we used the upper triangularity of $\overline{U}_s$.

We now expand the resolvent to rewrite the transition $\langle\tilde{\beta}_j|1/(z-\overline{U}_s)|\mathcal{B}_M\rangle$ as

$$\langle\tilde{\beta}_j|\frac{1}{z-\overline{U}_s}|\mathcal{B}_M\rangle=\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\langle\tilde{\beta}_j|\overline{U}_s^{n}|\mathcal{B}_M\rangle\ . \qquad (3.15)$$

Once $|\mathcal{B}_M\rangle$ makes a transition to some given intermediate state $|\beta_{j'}\rangle$, it never returns, due to the upper triangularity of $\overline{U}_s$. This transition may occur at any of $n-1$ possible places in $\langle\tilde{\beta}_j|\overline{U}_s^n|\mathcal{B}_M\rangle$. Using this fact and summing over all possible intermediate states, we have that

$$\langle\tilde{\beta}_j|\frac{1}{z-\overline{U}_s}|\mathcal{B}_M\rangle=-\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\sum_{j'=0}^{M}\sum_{l=1}^{n}\langle\tilde{\beta}_j|\overline{U}_s^{n-l}|\beta_{j'}\rangle\langle\tilde{\beta}_{j'}|\overline{U}_s|\mathcal{B}_M\rangle\langle \mathrm{e}\tilde{\beta}_M|\overline{U}_s^{l-1}|\mathcal{B}_M\rangle$$
$$=-\sum_{j'=j}^{M}\langle\tilde{\beta}_j|\frac{1}{z-\overline{U}_s}|\beta_{j'}\rangle\langle\tilde{\beta}_{j'}|\overline{U}_s|\mathcal{B}_M\rangle\langle \mathrm{e}\tilde{\beta}_M|\frac{1}{z-\overline{U}_s}|\mathcal{B}_M\rangle\ . \qquad (3.16)$$

Using this in (3.14) and by the completeness relation (2.20) that

$$\langle \mathrm{e}\tilde{\beta}_M|\frac{1}{z-\overline{U}_s}|\mathcal{B}_M\rangle\langle \mathrm{e}\tilde{\beta}_M|\rho_s\rangle=-\langle \mathrm{e}\tilde{\beta}_M|\frac{1}{z-\overline{U}_s}|\rho_s\rangle\ , \qquad (3.17)$$

we obtain

$$\frac{1}{z-\overline{U}_s}\rho_s(x,0)=\sum_{j'\ge j=0}^{M}\beta_j(x)\langle\tilde{\beta}_j|\frac{1}{z-\overline{U}_s}|\beta_{j'}\rangle\left\{\langle\tilde{\beta}_{j'}|\rho_s\rangle-\langle\tilde{\beta}_{j'}|\overline{U}_s|\mathcal{B}_M\rangle\langle \mathrm{e}\tilde{\beta}_M|\frac{1}{z-\overline{U}_s}|\rho_s\rangle\right\}$$
$$-\mathcal{B}_M(x)\langle \mathrm{e}\tilde{\beta}_M|\frac{1}{z-\overline{U}_s}|\rho_s\rangle\ . \qquad (3.18)$$

For a density that is at least M-times differentiable, the intertwining relation (3.12) gives

$$\langle \mathrm{e}\tilde{\beta}_M|\frac{1}{z-\overline{U}_s}|\rho_s\rangle=\langle \mathrm{e}\tilde{\beta}_0|\frac{1}{z-(\overline{U}_s/2^M)}|\frac{d^M}{dx^M}\rho_s\rangle\ , \qquad (3.19)$$

showing that $\langle \mathrm{e}\tilde{\beta}_M|1/(z-\overline{U}_s)|\rho_s\rangle$ is regular with respect to z for $|z|>(\frac{1}{2})^M$. By (3.18), then, the singularities of the resolvent (i.e., the spectrum) in the region $|z|>(\frac{1}{2})^M$ are determined by evaluating those of $\langle\tilde{\beta}_j|1/(z-\overline{U}_s)|\beta_{j'}\rangle$.

From the decomposition (3.11), we may write the "perturbation" expansion

$$\frac{1}{z-\overline{U}_s}=\sum_{n=0}^{\infty}\frac{1}{z-\overline{U}_{s0}}\left[\delta\overline{U}_s\frac{1}{z-\overline{U}_{s0}}\right]^n\ . \qquad (3.20)$$

Using this, repeatedly inserting the Euler-Maclaurin expansion, and using the upper triangularity of $\delta\overline{U}_s$ gives

$$\langle\tilde{\beta}_j|\frac{1}{z-\bar{U}_s}|\beta_{j'}\rangle=\sum_{n=0}^{\infty}\langle\tilde{\beta}_j|\frac{1}{z-\bar{U}_{s0}}\left[\delta\bar{U}_s\frac{1}{z-\bar{U}_{s0}}\right]^n|\beta_{j'}\rangle$$

$$=\frac{1}{z-e^{-\gamma_s^{(j)}}}\left\{\delta_{j,j'}+\sum_{n=1}^{\infty}\sum_{j_1,j_2,\ldots,j_{n-1}}\langle\tilde{\beta}_j|\delta\bar{U}_s|\beta_{j_1}\rangle\frac{1}{z-e^{-\gamma_s^{(j_1)}}}\langle\tilde{\beta}_{j_1}|\delta\bar{U}_s|\beta_{j_2}\rangle\cdots\frac{1}{z-e^{-\gamma_s^{(j_{n-1})}}}\times\langle\tilde{\beta}_{j_{n-1}}|\delta\bar{U}_s|\beta_{j'}\rangle\frac{1}{z-e^{-\gamma_s^{(j')}}}\right\}. \quad (3.21)$$

Due to the zero diagonal and upper triangularity of $\delta\bar{U}_s$, $j<j_1<j_2<\cdots<j'$, so that the "perturbation" expansion terminates at the $(j'-j)$th order. Thus, $\langle\tilde{\beta}_j|1/(z-\bar{U}_s)|\beta_{j'}\rangle$, has only simple poles at $z=e^{-\gamma_s^{(m)}}$ for $j\le m\le j'$ and no other singularities. The location of the poles are the eigenvalues. We note that for $s<L/2$, they are positive and for $s>L/2$, they are negative. For $s=L/2$, the eigenvalue is zero and the diagonal part of $\bar{U}_s$ as we have defined it in (3.11b) vanishes. The eigenmodes then belong to the null space.

Due to the simple analytic structure of the resolvent, we may shrink the contour in (3.13) and pick up the contributions from each of the determined poles. This means that the modes which decay slower then $(\frac{1}{2})^{Mt}$ are uniquely determined for

$$(d^M/dx^M)\rho_s(x,0)\in L_2 .$$

The smoothness of the initial probability density is a physical condition that specifies which decaying modes are realized [19,21].

The number of poles in the region $(\frac{1}{2})^M<|z|\le 1$ depends on the position of the first pole at $z=e^{-\gamma_s^{(0)}}$. Its location is given by

$$\frac{1}{2^{M_s}}<|e^{-\gamma_s^{(0)}}|\le\frac{2}{2^{M_s}} , \quad (3.22a)$$

where M_s is an integer greater than or equal to 1. The pole at $z=e^{-\gamma_s^{(m)}}$ thus satisfies

$$\frac{1}{2^{M_s+m}}<|e^{-\gamma_s^{(m)}}|\le\frac{2}{2^{M_s+m}} , \quad (3.22b)$$

so that there are $M-M_s+1$ poles in the region $(\frac{1}{2})^M<|z|\le 1$.

The time evolution of the density can now be written as a sum of contributions from each pole and a background integral whose contour is just outside of a circle whose radius is $(\frac{1}{2})^M$ as

$$\rho_s(x,t)=\sum_{m=0}^{M-M_s}\rho_s^{(m)}(x,t)+\bar{\mathcal{R}}_s^{(M)}(x,t) , \quad (3.23)$$

where

$$\rho_s^{(m)}(x,t)=\frac{1}{2\pi i}\oint_{z=e^{-\gamma_s^{(m)}}}dz\,z^{2t}\frac{1}{z-\bar{U}_s}\rho_s(x,0) \quad (3.24)$$

and

$$\bar{\mathcal{R}}_s^{(M)}(x,t)=\frac{1}{2\pi i}\oint_{|z|=2^{-M+\epsilon}}dz\,z^{2t}\frac{1}{z-\bar{U}_s}\rho_s(x,0) . \quad (3.25)$$

In order to evaluate $\rho_s^{(m)}(x,t)$, we assume that

$$(d^{M_s+m}/dx^{M_s+m})\rho_s(x,0)\in L_2$$

and use the expansion (3.18) up to the mth order since the pole at $e^{-\gamma_s^{(m)}}$ is associated with $|\beta_m\rangle\langle\tilde{\beta}_m|$. We then obtain

$$\rho_s^{(m)}(x,t)=\frac{1}{2\pi i}\oint_{z=e^{-\gamma_s^{(m)}}}dz\,z^{2t}\sum_{j=0}^{m}\beta_j(x)\langle\tilde{\beta}_j|\frac{1}{z-\bar{U}_s}|\beta_m\rangle\left\{\langle\tilde{\beta}_m|\rho_s\rangle-\langle\tilde{\beta}_m|\delta\bar{U}_s|\mathcal{B}_m\rangle\langle e\tilde{\beta}_m|\frac{1}{z-\bar{U}_s}|\rho_s\rangle\right\} . \quad (3.26)$$

Using

$$\frac{1}{z-\bar{U}_s}=\left[1+\frac{1}{z-\bar{U}_s}\delta\bar{U}_s\right]\frac{1}{z-\bar{U}_{s0}} \quad (3.27)$$

gives then

$$\rho_s^{(m)}(x,t)=\frac{1}{2\pi i}\oint_{z=e^{-\gamma_s^{(m)}}}dz\,z^{2t}\sum_{j=0}^{m}\beta_j(x)\langle\tilde{\beta}_j|1+\frac{1}{z-\bar{U}_s}\delta\bar{U}_s|\beta_m\rangle\frac{1}{z-e^{-\gamma_s^{(m)}}}\times\left\{\langle\tilde{\beta}_m|\rho_s\rangle-\langle\tilde{\beta}_m|\delta\bar{U}_s|\mathcal{B}_m\rangle\langle e\tilde{\beta}_m|\frac{1}{z-\bar{U}_s}|\rho_s\rangle\right\} . \quad (3.28)$$

Because of the simple pole, $\rho_s^{(m)}(x,t)$ can be written in terms of right and left decaying eigenstates as

$$\rho_s^{(m)}(x,t)=\gamma_s^{(m)}(x)e^{-\gamma_s^{(m)}2t}\langle\tilde{\gamma}_s^{(m)}|\rho_s\rangle\ , \tag{3.29}$$

where

$$\gamma_s^{(m)}(x)=\frac{1}{2\pi i}\oint_{z=e^{-\gamma_s^{(m)}}}dz\frac{1}{z-\overline{U}_s}\beta_m(x)$$
$$=\sum_{j=0}^{m}\beta_j(x)\langle\tilde{\beta}_j|1+\frac{1}{e^{-\gamma_s^{(m)}}-\overline{U}_s}\delta\overline{U}_s|\beta_m\rangle \tag{3.30}$$

and

$$\langle\tilde{\gamma}_s^{(m)}|\rho_s\rangle=\frac{1}{2\pi i}\oint_{z=e^{-\gamma_s^{(m)}}}dz\langle\tilde{\beta}_m|\frac{1}{z-\overline{U}_s}|\rho_s\rangle$$
$$=\langle\tilde{\beta}_m|\rho_s\rangle-\langle\tilde{\beta}_m|\delta\overline{U}_s|\mathcal{B}_m\rangle$$
$$\times\langle e\tilde{\beta}_m|\frac{1}{e^{-\gamma_s^{(m)}}-\overline{U}_s}|\rho_s\rangle\ . \tag{3.31}$$

Since the resolvent operator $1/(z-U)$ is well defined only if $|z|>\|U\|$, the resolvent operators in (3.30) and (3.31) are formal expressions ($|e^{-\gamma_s^{(m)}}|\le\|\overline{U}_s\|$). Hereafter we always interpret the resolvent operator inside the inner product as

$$\langle b|\frac{1}{z-U}|a\rangle\equiv\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\langle b|U^n|a\rangle\ . \tag{3.32}$$

The expressions for the eigenstates, (3.30) and (3.31), may be written in a more symmetric form by utilizing the completeness relation (2.20) and the upper triangularity of $\overline{U}_s$. We obtain then

$$\gamma_s^{(m)}(x)=\left[1+\frac{1}{e^{-\gamma_s^{(m)}}-\overline{U}_s}\delta\overline{U}_s\right]\beta_m(x) \tag{3.33}$$

and

$$\langle\tilde{\gamma}_s^{(m)}|\rho_s\rangle=\langle\tilde{\beta}_m|1+\frac{\delta\overline{U}_s}{e^{-\gamma_s^{(m)}}-\overline{U}_s}|\rho_s\rangle\ . \tag{3.34}$$

B. The right eigenstates

From the expression for the right eigenstates (3.30), we can write the following recursion relation:

$$\gamma_s^{(m)}(x)=\beta_m(x)+\sum_{j=0}^{m-1}\beta_j(x)\frac{1}{e^{-\gamma_s^{(m)}}-e^{-\gamma_s^{(j)}}}$$
$$\times\langle\tilde{\beta}_j|\delta\overline{U}_s|\gamma_s^{(m)}\rangle\ , \tag{3.35}$$

from which it is clear that $\gamma_s^{(m)}(x)$ is a polynomial in x of degree m. Using this expression and the matrix elements of $\overline{U}_s$ given in (A4), it is straightforward to calculate explicitly the right eigenstates.

The first four eigenstates are

$$\gamma_s^{(0)}(x)=1\ , \tag{3.36a}$$

$$\gamma_s^{(1)}(x)=(x-\tfrac{1}{2})+\tfrac{1}{2}i\tan\left[\frac{\pi s}{L}\right]\ , \tag{3.36b}$$

$$\gamma_s^{(2)}(x)=\tfrac{1}{2}(x^2-x+\tfrac{1}{6})+\tfrac{1}{2}(x-\tfrac{1}{2})i\tan\left[\frac{\pi s}{L}\right]$$
$$+\tfrac{1}{6}i^2\tan^2\left[\frac{\pi s}{L}\right]\ , \tag{3.36c}$$

$$\gamma_s^{(3)}(x)=\tfrac{1}{6}(x^3-\tfrac{3}{2}x^2+\tfrac{1}{2}x)$$
$$+\tfrac{1}{4}(x^2-x-\tfrac{1}{7})i\tan\left[\frac{\pi s}{L}\right]$$
$$+\tfrac{1}{6}(x-\tfrac{1}{2})i^2\tan^2\left[\frac{\pi s}{L}\right]$$
$$+\tfrac{1}{21}i^3\tan^3\left[\frac{\pi s}{L}\right]\ . \tag{3.36d}$$

The eigenstates satisfy the relation

$$d\gamma_s^{(m)}(x)/dx=\gamma_s^{(m-1)}(x)\ .$$

The eigenstates of the full multi-Bernoulli map,

$$\Gamma_s^{(m)}(X)=\Gamma_s^{(m)}(q+x)\ ,$$

are related to the states of the transformed map by

$$\Gamma_s^{(m)}(q+x)=\frac{1}{\sqrt{L}}e^{i(2\pi/L)qs}\gamma_s^{(m)}(x)\ , \tag{3.37}$$

with the same relation holding for the left eigenstates, $\tilde{\Gamma}_s^{(m)}(q+x)$. [The subscript s in $\Gamma_s^{(m)}(X)$ is a label for the eigenstate and does not denote the transform of the undefined object $\Gamma^{(m)}(X)$.] We may construct pairs of real eigenstates, $\hat{\Gamma}_s^{(m)}(q+x)$ and $\check{\Gamma}_s^{(m)}(q+x)$, by taking linear combinations of states with the same eigenvalue as

$$\hat{\Gamma}_s^{(m)}(q+x)\equiv\tfrac{1}{2}[\Gamma_s^{(m)}(q+x)+\Gamma_{L-s}^{(m)}(q+x)] \tag{3.38a}$$

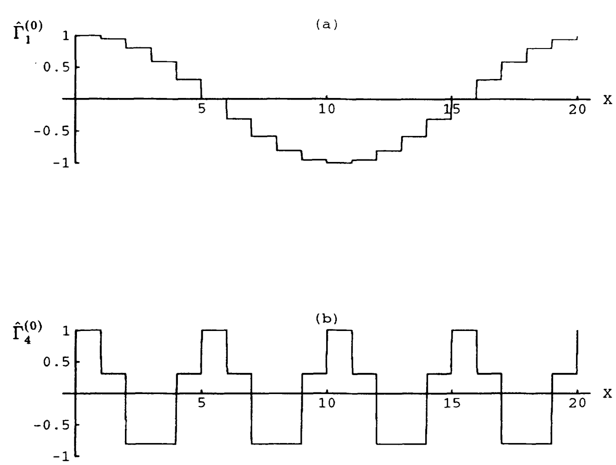

FIG. 2. The right eigenstates: (a) $\hat{\Gamma}_1^{(0)}(X)$ and (b) $\hat{\Gamma}_4^{(0)}(X)$.

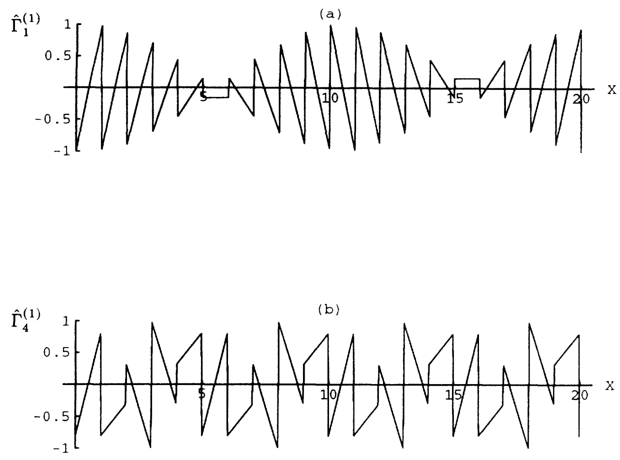

FIG. 3. The right eigenstates: (a) $\hat{\Gamma}_1^{(1)}(X)$ and (b) $\hat{\Gamma}_4^{(1)}(X)$.

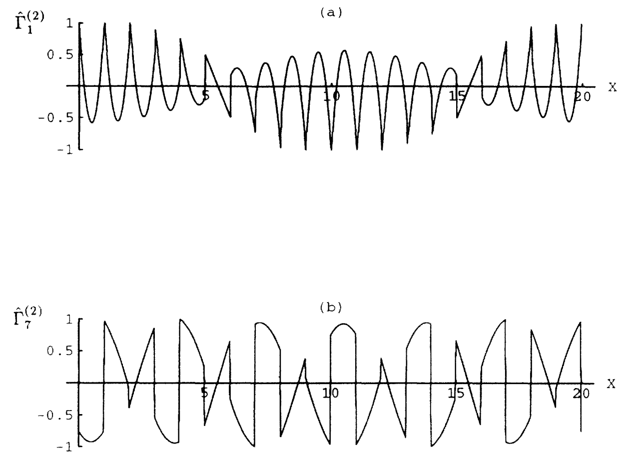

FIG. 4. The right eigenstates: (a) $\hat{\Gamma}_1^{(2)}(X)$ and (b) $\hat{\Gamma}_7^{(2)}(X)$.

and

$$\check{\Gamma}_s^{(m)}(q+x) \equiv \frac{1}{2i}[\Gamma_s^{(m)}(q+x) - \Gamma_{L-s}^{(m)}(q+x)] . \tag{3.38b}$$

The explicit forms of $\hat{\Gamma}_s^{(m)}(q+x)$ for $m=0$, 1, and 2 are

$$\hat{\Gamma}_s^{(0)}(q+x) = \frac{1}{\sqrt{L}} \cos\left[\frac{2\pi}{L}qs\right] , \tag{3.39}$$

$$\hat{\Gamma}_s^{(1)}(q+x) = \frac{1}{\sqrt{L}}\left[(x-\tfrac{1}{2})\cos\left[\frac{2\pi}{L}qs\right] - \tfrac{1}{2}\tan\left[\frac{\pi s}{L}\right]\sin\left[\frac{2\pi}{L}qs\right]\right] , \tag{3.40}$$

$$\hat{\Gamma}_s^{(2)}(q+x) = \frac{1}{\sqrt{L}}\left\{\left[\tfrac{1}{2}(x^2-x+\tfrac{1}{6}) - \tfrac{1}{6}\tan^2\left[\frac{\pi s}{L}\right]\right] \times \cos\left[\frac{2\pi}{L}qs\right] - \tfrac{1}{2}(x-\tfrac{1}{2})\tan\left[\frac{\pi s}{L}\right]\sin\left[\frac{2\pi}{L}qs\right]\right\} . \tag{3.41}$$

Graphs of right eigenstates $\hat{\Gamma}_s^{(m)}(q+x)$ for some representative values of m and s are given in Figs. 2–4. We may obtain $\check{\Gamma}_s^{(m)}(q+x)$ from the above expressions for $\hat{\Gamma}_s^{(m)}(q+x)$ by changing the phase $(2\pi/L)qs$ to $(2\pi/L)qs - \pi/2$.

C. The exact kinetic dynamics

The eigenvalues of the full map, $e^{-\Gamma_s^{(m)}}$ are the squares of those of the transformed map, i.e.,

$$e^{-\Gamma_s^{(m)}} = e^{-2\gamma_s^{(m)}} , \tag{3.42}$$

so that the decay rates $\Gamma_s^{(m)}$ are

$$\Gamma_s^{(m)} = m\ \ln 4 - 2\ln\left|\cos\left[\frac{\pi s}{L}\right]\right| . \tag{3.43}$$

The term $m \ln 4$ comes from the intercell dynamics governed by a 4-adic map. It gives a quick approach (for $m>0$) to local equilibrium inside the cells. The term $-2\ln|\cos(\pi s/L)|$ is due to the intracell dynamics and gives a slow approach to global equilibrium through diffusion.

The long time dynamics of the system is governed by the slowest decaying mode which is the $m=0$ mode with eigenvalue $e^{-\Gamma_s^{(0)}}$, where

$$\Gamma_s^{(0)} = -2\ln\left|\cos\left[\frac{\pi s}{L}\right]\right| = \frac{1}{4}\left[\frac{2\pi s}{L}\right]^2 + \frac{1}{96}\left[\frac{2\pi s}{L}\right]^4 + \cdots . \tag{3.44}$$

Since $2\pi s/L$ is conjugate to q, we have $2\pi s/L \sim -i(\partial/\partial q)$. The time evolution of the state $\Gamma_s^{(0)}$ is then governed by the diffusion equation

$$(\partial/\partial t)\Gamma_s^{(0)} = D(\partial^2/\partial q^2)\Gamma_s^{(0)}$$

as

$$\bar{U}_{mB}^t \Gamma_s^{(0)}(q+x) \sim e^{-(1/4)(2\pi s/L)^2 t}\Gamma_s^{(0)}(q+x) \sim e^{(1/4)\partial^2/\partial q^2 t}\Gamma_s^{(0)}(q+x) . \tag{3.45}$$

Thus, $\Gamma_s^{(0)}(q+x)$ is the eigenstate associated with diffusion and the diffusion coefficient $D=\frac{1}{4}$. Since we have the exact dispersion relation (3.44), we may also obtain the higher order diffusion coefficients, such as the Burnett coefficient $B=\frac{1}{96}$ obtained by Gaspard [20].

We may consider the mean square deviation of q with respect to the mth mode, $\langle q^2 \rangle_{(m)}$. Neglecting boundary effects and considering an initial distribution finite in the $q=0$ cell only (with the configuration of the system centered at $X=0$), we obtain

$$\langle q^2 \rangle_{(m)} = \frac{t}{2} e^{-\Gamma_0^{(m)} t} = \frac{2Dt}{4^{mt}} , \tag{3.46}$$

showing explicitly that for short times all modes contribute to the mean square deviation but that the contributions of all modes, except for the $m=0$ mode, decay with rates that are multiples of the Lyapunov time. The $m=0$ mode, corresponding to diffusion, shows purely linear growth of the mean squared deviation.

D. The left eigenstates

The left eigenstates as given in (3.31) act as functionals on a density belonging to the test space of suitable functions. Before proceeding with the mathematical analysis, it is worthwhile to consider what structure we may expect for the left eigenstates. From (3.10) the adjoint of the Frobenius-Perron operator is

$$\overline{U}_s^\dagger = e^{-i(\pi s/L) r_1(x)} \overline{U}_B^\dagger \; . \tag{3.47}$$

The eigenvalue equation

$$\overline{U}_s^\dagger \tilde{\gamma}_s^{(m)}(x) = e^{-\gamma_s^{(m)}} \tilde{\gamma}_s^{(m)}(x)$$

can be rewritten as

$$\overline{U}_B^\dagger \tilde{\gamma}_s^{(m)}(x) = e^{-\gamma_s^{(m)} + i(\pi s/L) r_1(x)} \tilde{\gamma}_s^{(m)}(x) \; . \tag{3.48}$$

The adjoint of the Frobenius-Perron operator of the Bernoulli map, $\overline{U}_B^\dagger$, acts essentially as the scale transformation $x \to 2x$. Since the eigenstates are invariant (except for a phase factor) under this scale transformation, they must obviously have a self-similar nature. A noninteger dimension is expected for $\tilde{\gamma}_s^{(m)}(x)$ when $s>0$ since then the eigenvalue $e^{-\gamma_s^{(m)}}$ is not an integer power of the scaling factor of 2.

Without loss of generality, we may consider in detail only the $m=0$ left eigenstate, $\tilde{\gamma}_s^{(0)}(x)$, since the action of the mth left eigenstate is given simply in terms of the $m=0$ eigenstate as

$$\langle \tilde{\gamma}_s^{(m)} | \rho_s \rangle = \langle \tilde{\gamma}_s^{(0)} | \frac{d^m}{dx^m} \rho_s \rangle \; . \tag{3.49}$$

We are mainly interested in the long time behavior of the probability density so we will consider the case of $M_s=1$. Then for $(d/dx)\rho_s(x,0) \in L_2$ [for $M_s=2$ we would require here that $(d^2/dx^2)\rho_s(x,0) \in L_2$], we have by going to the next order of the Euler-Maclaurin expansion of (3.31) that

$$\begin{aligned}\langle \tilde{\gamma}_s^{(0)} | \rho_s \rangle &= \langle \tilde{\beta}_0 | \rho_s \rangle + \langle \tilde{\beta}_0 | \delta \overline{U}_s \{ |\beta_1\rangle\langle \tilde{\beta}_1| - |\mathcal{B}_1\rangle\langle e\tilde{\beta}_1| \} \frac{1}{e^{-\gamma_s^{(0)}} - \overline{U}_s} | \rho_s \rangle \\ &= \langle \tilde{\beta}_0 | \rho_s \rangle + e^{-\gamma_s^{(0)}} i \tan\left[\frac{\pi s}{L}\right] \int_0^1 dx\, dx' r_1(x) [\beta_1(x) - \beta_1(x-x')] \frac{1}{e^{-\gamma_s^{(0)}} - \overline{U}_s/2} \frac{d}{dx'} \rho_s(x') \; .\end{aligned} \tag{3.50}$$

Performing the integration over x gives

$$-\int_0^1 dx\, r_1(x)[\beta_1(x) - \beta_1(x-x')] = \lambda_1(x') \equiv \begin{cases} x' & \text{if } x' < \frac{1}{2} \\ 1-x' & \text{if } x' \geq \frac{1}{2} \; , \end{cases} \tag{3.51}$$

where the argument of $\lambda_1(x)$ is taken modulo 1. Thus,

$$\langle \tilde{\gamma}_s^{(0)} | \rho_s \rangle = \langle \tilde{\beta}_0 | \rho_s \rangle - i \tan\left[\frac{\pi s}{L}\right] \langle w_{s,\lambda_1} | \frac{d\rho_s}{dx} \rangle \; , \tag{3.52}$$

where

$$w_{s,f}(x) \equiv \frac{1}{1 - \overline{U}_s^\dagger e^{\gamma_s^{(0)}}/2} f(x) \; , \tag{3.53}$$

whose inner product, when $|e^{\gamma_s^{(0)}}/2| < 1$, with any L_1 function is well defined. Under the scale transformation given by $\overline{U}_B^\dagger$, the function $w_{s,f}(x)$ satisfies

$$\overline{U}_B^\dagger w_{s,f}(x) = 2 e^{-\gamma_s^{(0)} - i(\pi s/L) r_1(x)} [w_{s,f}(x) - f(x)] \; . \tag{3.54}$$

We can give an explicit form of (3.53) by doing a series expansion of the term with $\overline{U}_s^\dagger$ as

$$w_{s,\lambda_1}(x) = \sum_{n=0}^{\infty} \left[\frac{\overline{U}_s^\dagger e^{\gamma_s^{(0)}}}{2}\right]^n \lambda_1(x) \; . \tag{3.55}$$

From the form of $\overline{U}_s^\dagger$, (3.47), and using that $(\overline{U}_B^\dagger)^n r_1(x) = r_{n+1}(x)$ gives for the nth power of $\overline{U}_s^\dagger$,

$$(\overline{U}_s^\dagger)^n = e^{-i(\pi s/L) \sum_{j=1}^{n} r_j(x)} (\overline{U}_B^\dagger)^n \; . \tag{3.56}$$

Using this in (3.55) gives $w_{s,\lambda_1}(x)$ as

$$w_{s,\lambda_1}(x) = \sum_{n=0}^{\infty} \frac{\exp\left[-i\left[\frac{\pi s}{L}\right] \sum_{j=1}^{n} r_j(x)\right]}{\left[\cos\left[\frac{\pi s}{L}\right]\right]^n} \frac{\lambda_1(2^n x)}{2^n} \; . \tag{3.57}$$

The self-similar nature of $w_{s,\lambda_1}(x)$ is clear since it is made up of rescaled copies of $\lambda_1(x)$ multiplied by a somewhat complicated x-dependent factor involving the complex exponential of a sum of Rademacher functions. The function $w_{s,\lambda_1}(x)$ is nowhere differentiable and except for $s=0$ is discontinuous.

In the limit of $s \to 0$, we have

$$w_{s=0,\lambda_1}(x) = \sum_{n=0}^{\infty} \frac{\lambda_1(2^n x)}{2^n} = 2T(x) \; , \tag{3.58}$$

where $T(x)$ is the continuous but nowhere differentiable Takagi function [35]. The graph of the curve $T(x)$ is known [36] to have Hausdorff dimension, $D_H[T(x)]=1$, so that $w_{s=0,\lambda_1}(x)$ has Hausdorff dimension of 1 also. For s/L increasing from 0 to $\frac{1}{3}$, the radius of convergence of (3.55), numerical evidence leads us to conjecture that the graph of (3.57) has Hausdorff dimension increasing from 1 to 2. In Fig. 5 the real part of $w_{s,\lambda_1}(x)$ is represented for three values of s/L.

For $s=0$ we recover the result of the Bernoulli map as $\tilde{\gamma}_0^{(m)}(x)=d^m/dx^m$. For $s>0$ the transformation corresponding to $\overline{U}_s$ is no longer a measure preserving transformation with respect to Lebesgue measure. This is clear since then $e^{-\gamma_s^{(0)}}<1$. Thus there is an escape of "probability" and the dynamics of $\overline{U}_s^\dagger$ settles onto an invariant fractal set. Thus the eigenstates of $\overline{U}_s^\dagger$ (left eigenstates of $\overline{U}_s$) which are invariant (except for some numerical factor) should have the same fractal structure [37].

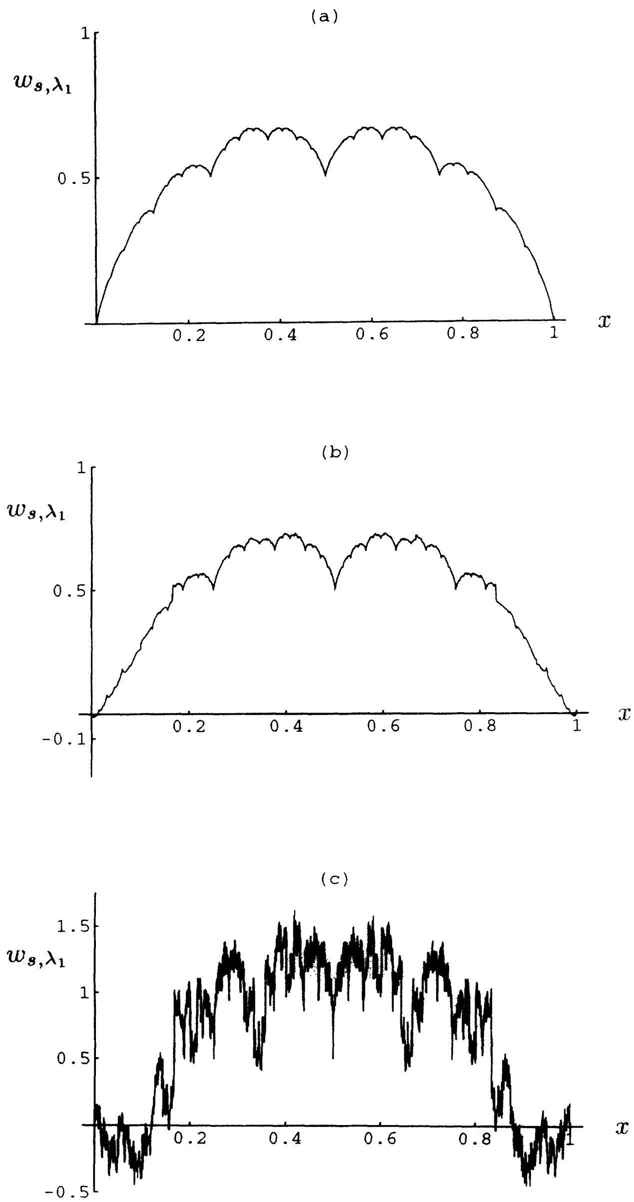

FIG. 5. The real part of the function $w_{s,\lambda_1}(x)$ for (a) $s/L=0$, (b) $s/L=0.15$, and (c) $s/L=0.3$.

If we assume that ρ_s is infinitely differentiable, then we may extend the expansion in (3.31) to infinity. We recover then the form of the left eigenstate given by Gaspard [20].

IV. THE BAKER TRANSFORMATION

In the previous sections we have studied one-dimensional and hence noninvertible chaotic maps. The Frobenius-Perron operators for these maps were not unitary, but their adjoints were isometric which forced us to go out of the Hilbert space to construct a complete spectral representation including decaying modes. In this section we study the invertible two-dimensional baker map. Its Frobenius-Perron operator is unitary so that now both the left as well as the right decaying eigenstates will belong to generalized functional spaces. We will see that physical conditions for obtaining the generalized representation are necessary for both the observable and the probability density. We will thus consider the evolution of correlation functions.

For the baker transformation the poles of the resolvent operator are degenerate. This is due to the fact that the system is area preserving so that there are pairs of both positive and negative Lyapunov exponents. The Ruelle-Pollicott resonances of the baker transformation are directly related to its Lypounov exponents [2]. As will be shown, the mth pole has $m+1$ degeneracy. Thus, we need to construct an $m+1$ dimensional eigenspace instead of just a simple eigenstate. This leads to a time evolution that is not a sum of pure exponentially decaying terms but now the exponential decay is modified by factors which are polynomials in t. In order to calculate the eigenspace systematically, we introduce a projective decomposition of the resolvent.

The baker transformation is a one-to-one transformation on the unit square given by

$$(x_{n+1},y_{n+1})=F(x_n,y_n)$$
$$=\begin{cases}(2x_n,y_n/2) & \text{if } 0\le x_n<\frac{1}{2}\\ (2x_n-1,y_n/2+\frac{1}{2}) & \text{if } \frac{1}{2}\le x_n<1\ .\end{cases} \quad (4.1)$$

Since the map is invertible and its Jacobian is 1, the action of the Frobenius-Perron operator U_b is simply given by $U_b A(x,y)=A[F^{-1}(x,y)]$. Specifically,

$$U_b A(x,y)=\begin{cases}A(x/2,2y) & \text{if } 0\le y<\frac{1}{2}\\ A(x/2+\frac{1}{2},2y-1) & \text{if } \frac{1}{2}\le y<1\ .\end{cases} \quad (4.2)$$

Since the map reduces to the Bernoulli map if we neglect y, the operator U_b satisfies the following intertwining relation analogous to (2.21):

$$\frac{\partial^M}{\partial x^M}U_b^n A(x,y)=\left[\frac{1}{2^n}\right]^M U_b^n\frac{\partial^M}{\partial x^M}A(x,y)\ , \quad (4.3)$$

if $A(x,y)$ is at least M-times differentiable with respect to x. Similarly, the adjoint operator $U_b^\dagger$ satisfies

$$\frac{\partial^M}{\partial y^M}(U_b^\dagger)^n B(x,y)=\left[\frac{1}{2^n}\right]^M (U_b^\dagger)^n \frac{\partial^M}{\partial y^M}B(x,y)\ , \quad (4.4)$$

if $B(x,y)$ is at least M-times differentiable with respect to y.

A two-dimensional Bernoulli basis is introduced through the Euler-Maclurin expansion with respect to x for the right state and with respect to y for the left state. For example, the inner product of $A(x,y)$ and $B(x,y)$ is expanded as

$$(B|A)=\sum_{i,j=0}^{M}[(B|\beta_i,\tilde\beta_j)(\tilde\beta_i,\beta_j|A)-\delta_{i,M}(B|\mathcal{B}_M,\tilde\beta_j)(e\tilde\beta_M,\beta_j|A) \\ -\delta_{j,M}(B|\beta_i,\tilde\beta_M e)(\tilde\beta_i,\mathcal{B}_M|A)+\delta_{i,M}\delta_{j,M}(B|\mathcal{B}_M,\tilde\beta_M e)(e\tilde\beta_M,\mathcal{B}_M|A)]\ , \quad (4.5)$$

where we used the following two-dimensional bra-ket notation:

$$(B|A)\equiv\int_0^1 dx\,dy\,B^*(x,y)A(x,y)\ . \quad (4.6)$$

We note that, for example, the generalized function $\tilde\beta_j(y)$ in (4.5) can be interpreted as the j-times differential operator of y which operates to the left.

The Frobenius-Perron operator of the baker transformation (4.2) may be rewritten as

$$U_b A(x,y)=\bar U_x[1+r_1(x)r_1(y)]\bar U_y^\dagger A(x,y)\ , \quad (4.7)$$

where $\bar U_x$ is the Frobenius-Perron operator of the Bernoulli map acting on the x coordinate, $\bar U_y^\dagger$ is the Koopman operator of the Bernoulli map acting on the y coordinate, and r_1 is again the first Rademacher function. The decomposition (4.7) splits the Frobenius-Perron operator into a part diagonal, $U_{b0}=\bar U_x\bar U_y^\dagger$, with respect to the two-dimensional Bernoulli basis, and an off-diagonal part,

$$\delta U_b=\bar U_x r_1(x)r_1(y)\bar U_y^\dagger\ .$$

The matrix elements of U_b with respect to the two-dimensional basis are given in Appendix B. They are also triangular, as for the multi-Bernoulli map.

A. Time evolution of correlation functions

The time evolution of correlation functions under the baker transformation has recently been studied by several authors [2,38]. Although they restricted the observable and probability density as a polynomial [2] or a certain class of analytic functions [38], there is no physical reason to consider such a strong restriction. In this section we construct the correlation with only the restriction that the observable and density be m-times differentiable.

We consider the time evolution of the correlation function of the observable $B(x,y)$ and probability density $A(x,y)$, i.e., $(B|U_b^t|A)$. Using the resolvent formalism, we have

$$(B|U_b^t|A)=\frac{1}{2\pi i}\oint_{|z|=1+\epsilon}dz\,z^t(B|\frac{1}{z-U_b}|A)\ . \quad (4.8)$$

For a smooth probability density with respect to the dilating direction and a smooth observable with respect to the contracting direction, i.e., $\partial_x^M A,\partial_y^M B\in L_2$, we can rewrite the resolvent $(B|1/(z-U_b)|A)$ using the Euler-Maclaurin expansions in (4.5):

$$(B|\frac{1}{z-U_b}|A)=(R_B^{(M)}|z-U_b|R_A^{(M)})+\sum_{i=0}^{M}(R_B^{(M)}|\beta_i,\tilde\beta_M e)(\tilde\beta_i,\mathcal{B}_M|A+U_bR_A^{(M)}) \\ +\sum_{j=0}^{M}(B+U_b^\dagger R_B^{(M)}|\mathcal{B}_M,\tilde\beta_j)(e\tilde\beta_M,\beta_j|R_A^{(M)}) \\ +\sum_{i'\ge i=0}^{M}\sum_{j\ge j'=0}^{M}(B+U_b^\dagger R_B^{(M)}|\beta_i,\tilde\beta_j)(\tilde\beta_i,\beta_j|\frac{1}{z-U_b}|\beta_{i'},\tilde\beta_{j'})(\tilde\beta_{i'},\beta_{j'}|A+U_bR_A^{(M)})\ , \quad (4.9)$$

where

$$R_A^{(M)}(x,y)\equiv-\int_0^1 dx'\mathcal{B}_M(x)\cdot e^*(x')\tilde\beta_M(x')\frac{1}{z-U_b}A(x',y)\ , \quad (4.10a)$$

$$R_B^{(M)}(x,y)\equiv-\int_0^1 dy'\mathcal{B}_M(y)\cdot e^*(y')\tilde\beta_M(y')\frac{1}{z-U_b^\dagger}B(x,y')\ . \quad (4.10b)$$

The derivation of (4.9) is given in Appendix C.

From the intertwining relations (4.3) and (4.4), it is easy to show that the resolvents $R_A^{(M)}(x,y)$ and $R_B^{(M)}(x,y)$ are regular with respect to z for $|z|>(\frac{1}{2})^M$ as

$$R_A^{(M)}(x,y)=-\int_0^1 dx'\mathcal{B}_M(x)\cdot\mathbf{e}^*(x')\frac{1}{z-U_b/2^M}\partial_{x'}^M A(x',y)\ , \tag{4.11a}$$

$$R_B^{(M)}(x,y)=-\int_0^1 dy'\mathcal{B}_M(y)\cdot\mathbf{e}^*(y')\frac{1}{z-U_b^\dagger/2^M}\partial_{y'}^M B(x,y')\ . \tag{4.11b}$$

Thus only the last of the four terms in (4.9) is singular for $|z|>(\frac{1}{2})^M$ and for smooth observables the singularities of the resolvent in this region are determined by evaluating those of $(\tilde{\beta}_i,\beta_j|1/(z-U_b)|\beta_{i'},\tilde{\beta}_{j'})$. Here, again, the smoothness is a "physical condition" that determines which decaying modes are realized in a physical representation of the time evolution. For the baker transformation it is necessary to consider not only the smoothness of the initial distribution function $A(x,y)$ with respect to the dilating direction, but also that of the final observable $B(x,y)$, with respect to the contracting direction.

The singularities of $(\tilde{\beta}_i,\beta_j|1/(z-U_b)|\beta_{i'},\tilde{\beta}_{j'})$ can be determined from the expansion analogous to (3.21):

$$\begin{aligned}(\tilde{\beta}_i,\beta_j|\frac{1}{z-U_b}|\beta_{i'},\tilde{\beta}_{j'})&=\sum_{n=0}^{\infty}(\tilde{\beta}_i,\beta_j|\frac{1}{z-U_{b0}}\left[\delta U_b\frac{1}{z-U_b}\right]^n|\beta_{i'},\tilde{\beta}_{j'})\\
&=\frac{1}{z-e^{-\gamma^{(i+j)}}}\Bigg\{\delta_{i,i'}\delta_{j,j'}+\sum_{n=1}^{\infty}\sum_{\substack{i_1,i_2,\ldots,i_{n-1}\\ j_1,j_2,\ldots,j_{n-1}}}(\tilde{\beta}_i,\beta_j|\delta U_b|\beta_{i_1}\tilde{\beta}_{j_1})\frac{1}{z-e^{-\gamma^{(i_1+j_1)}}}\\
&\qquad\times(\tilde{\beta}_{i_1}\beta_{j_1}|\delta U_b|\beta_{i_2}\tilde{\beta}_{j_2})\cdots\frac{1}{z-e^{-\gamma^{(i_{n-1}+j_{n-1})}}}\\
&\qquad\times(\tilde{\beta}_{i_{n-1}}\beta_{j_{n-1}}|\delta U_b|\beta_{i'}\tilde{\beta}_{j'})\frac{1}{z-e^{-\gamma^{(i'+j')}}}\Bigg\}\ .\end{aligned} \tag{4.12}$$

The matrix elements of δU_b are strictly upper triangular (zero diagonal). Thus intermediate states are ordered as $i<i_1<i_2<\cdots<i'$, and $j>j_1>j_2>\cdots>j'$, so that the "perturbation" expansion terminates at the order $\min\{i'-i,j-j'\}$. The singularities of $(\tilde{\beta}_i,\beta_j|1/(z-U_b)|\beta_{i'},\tilde{\beta}_{j'})$ are thus given by those of the resolvent of the diagonal part of the Frobenius-Perron operator and are poles at $z=e^{-\gamma^{(m)}}$ for $m=0,1,\ldots,i'+j$. Since i_1+j_1 can equal i_2+j_2 even for $i_1<i_2$ and $j_1>j_2$, the poles are degenerate in general.

The time correlation function is given as the sum of contributions from each of the poles and the background integral at $|z|=(\frac{1}{2})^M+\epsilon$ as

$$(B|U_b^t|A)=\frac{1}{2\pi i}\oint_{|z|=1+\epsilon}dz\,z^t(B|\frac{1}{z-U_b}|A)=\sum_{m=0}^{M-1}(B|\Sigma^{(m)}(t)|A)+(B|\mathcal{R}^{(M)}|A)\ , \tag{4.13}$$

where

$$(B|\Sigma^{(m)}(t)|A)=\frac{1}{2\pi i}\oint_{z=e^{-\gamma^{(m)}}}dz\,z^t(B|\frac{1}{z-U_b}|A) \tag{4.14}$$

and

$$(B|\mathcal{R}^{(M)}|A)=\frac{1}{2\pi i}\oint_{|z|=1/2^M+\epsilon}dz\,z^t(B|\frac{1}{z-U_b}|A)\ . \tag{4.15}$$

B. Projective decomposition of the resolvent

In order to evaluate the mth decaying mode, $(B|\Sigma^{(m)}(t)|A)$, we choose $M=m+1$ since it is the minimum condition necessary to specify the mth pole. By substituting (4.9) into (4.14) we obtain

$$\begin{aligned}(B|\Sigma^{(m)}(t)|A)=\frac{1}{2\pi i}\oint_{z=e^{-\gamma^{(m)}}}dz\,z^t\sum_{i'\ge i=0}^{m+1}\sum_{j\ge j'=0}^{m+1}&(B+U_b^\dagger R_B^{(m+1)}|\beta_i,\tilde{\beta}_j)(\tilde{\beta}_i,\beta_j|\frac{1}{z-U_b}|\beta_{i'},\tilde{\beta}_{j'})\\
&\times(\tilde{\beta}_{i'},\beta_{j'}|A+U_bR_A^{(m+1)})\ ,\end{aligned} \tag{4.16}$$

where the regular parts of (4.9) do not contribute after integration so they are not written here.

Because of the degeneracy, we introduce a projective decomposition [2,10,39] of the resolvent to isolate the poles. We define the generalized projection operators which include all the members of the two-dimensional Bernoulli basis whose eigenvalue of U_{b0} is $e^{-\gamma^{(m)}}$,

$$P^{(m)}=\sum_{i+j=m}|\beta_i,\tilde{\beta}_j)(\tilde{\beta}_i,\beta_j| , \quad (4.17a)$$

$$Q^{(m)}=I_M-P^{(m)} , \quad (4.17b)$$

where I_M is the identity operator. For such generalized projectors we have the following identity:

$$\frac{1}{z-U_b}=[P^{(m)}+\mathcal{C}^{(m)}(z)]\frac{1}{z-\Psi^{(m)}(z)}[P^{(m)}+\mathcal{D}^{(m)}(z)]+\mathcal{P}^{(m)}(z) , \quad (4.18)$$

where the operators $\mathcal{C}^{(m)}(z)$, $\mathcal{D}^{(m)}(z)$, $\mathcal{P}^{(m)}(z)$, and $\Psi^{(m)}$ are defined by

$$\mathcal{C}^{(m)}(z)\equiv\frac{1}{z-Q^{(m)}U_bQ^{(m)}}Q^{(m)}U_bP^{(m)} , \quad (4.19a)$$

$$\mathcal{D}^{(m)}(z)\equiv P^{(m)}U_bQ^{(m)}\frac{1}{z-Q^{(m)}U_bQ^{(m)}} , \quad (4.19b)$$

$$\mathcal{P}^{(m)}(z)\equiv Q^{(m)}\frac{1}{z-Q^{(m)}U_bQ^{(m)}}Q^{(m)} , \quad (4.19c)$$

$$\Psi^{(m)}(z)\equiv P^{(m)}U_bP^{(m)}+P^{(m)}U_bQ^{(m)}\mathcal{C}^{(m)}(z) . \quad (4.19d)$$

The proof of the identity (4.18) is given, for example, in [39].

Since $Q^{(m)}$ only includes members of the Bernoulli basis whose eigenvalue with operation by U_{b0} is not $e^{-\gamma^{(m)}}$, there is no pole such as $1/(z-e^{-\gamma^{(m)}})$ in the perturbation expansion of $1/(z-Q^{(m)}U_bQ^{(m)})$, i.e., $\mathcal{C}^{(m)}(z)$, $\mathcal{D}^{(m)}(z)$, and $\mathcal{P}^{(m)}(z)$ are regular at $z=e^{-\gamma^{(m)}}$. The upper triangularity of U_b guarantees that these expansions terminate. Hence, the projective decomposition decomposes the resolvent into the regular parts $\mathcal{C}^{(m)}(z)$, $\mathcal{D}^{(m)}(z)$, $\mathcal{P}^{(m)}(z)$, and $\Psi^{(m)}(z)$ and the singular part $1/[z-\Psi^{(m)}(z)]$ at $z=e^{-\gamma^{(m)}}$.

The operators $\mathcal{C}^{(m)}(z)$ and $\mathcal{D}^{(m)}(z)$ can be evaluated through the following recursion formulas, respectively:

$$\begin{aligned}(\tilde{\beta}_{m-j-k},\beta_{j+l}|\mathcal{C}^{(m)}(z)|\beta_{m-j},\tilde{\beta}_j)
&=\frac{1}{z-e^{-\gamma^{(m-k+l)}}}\Bigg\{(\tilde{\beta}_{m-j-k},\beta_{j+l}|U_b|\beta_{m-j},\tilde{\beta}_j)\\
&\quad+\sum_{(k',l')=0}^{(k,l)}(\tilde{\beta}_{m-j-k},\beta_{j+l}|U_b|\beta_{m-j-k'},\tilde{\beta}_{j+l'})(\tilde{\beta}_{m-j-k'},\beta_{j+l'}|\mathcal{C}^{(m)}(z)|\beta_{m-j},\tilde{\beta}_j)\Bigg\} , \end{aligned} \quad (4.20)$$

$$\begin{aligned}(\tilde{\beta}_{m-j},\beta_j|\mathcal{D}^{(m)}(z)|\beta_{m-j+k},\tilde{\beta}_{j-l})
&=\frac{1}{z-e^{-\gamma^{(m-k+l)}}}\Bigg\{(\tilde{\beta}_{m-j},\beta_j|U_b|\beta_{m-j+k},\tilde{\beta}_{j-l})\\
&\quad+\sum_{(k',l')=0}^{(k,l)}(\tilde{\beta}_{m-j},\beta_j|U_b|\beta_{m-j+k'},\tilde{\beta}_{j-l'})(\tilde{\beta}_{m-j+k'},\beta_{j-l'}|\mathcal{D}^{(m)}(z)|\beta_{m-j+k},\tilde{\beta}_{j-l})\Bigg\} , \end{aligned} \quad (4.21)$$

where the summation should be interpreted as

$$\sum_{(k',l')=0}^{(k,l)}\equiv\sum_{k'=0}^{k}\sum_{l'=0}^{l}(1-\delta_{k',l'})(1-\delta_{k,k'}\delta_{l,l'}) . \quad (4.22)$$

We now show how to evaluate the pole in $1/(z-\Psi^{(m)})$. The first term in the definition of $\Psi^{(m)}$ is explicitly

$$P^{(m)}U_bP^{(m)}=\sum_{i+j=m}\sum_{i'+j'=m}|\beta_i,\tilde{\beta}_j)(\tilde{\beta}_i,\beta_j|U_b|\beta_{i'},\tilde{\beta}_{j'})\times(\tilde{\beta}_{i'},\beta_{j'}| , \quad (4.23)$$

which is upper triangular with $e^{-\gamma^{(m)}}$ on the diagonal. For convenience we may consider it as an $(m+1)\times(m+1)$ matrix and keep in mind that it is an operator on the full space with nonzero entries only in the $P^{(m)}$ subspace. The second term can be calculated using the recursion formula (4.20) for $\mathcal{C}^{(m)}(z)$. It is a strictly upper-triangular square matrix with zeros on the diagonal. Thus, $\Psi^{(m)}$ is an upper triangular matrix with $e^{-\gamma^{(m)}}$ on the diagonal and the part above the diagonal we denote by $\Delta^{(m)}$. We have then

$$\Psi^{(m)}(z)=P^{(m)}e^{-\gamma^{(m)}}+\Delta^{(m)} , \quad (4.24)$$

where

$$\Delta^{(m)} \equiv P^{(m)}\delta U_b P^{(m)} + P^{(m)}\delta U_b \mathcal{C}^{(m)}(z) \ . \tag{4.25}$$

Since $\Delta^{(m)}$ is strictly upper triangular and of dimension $(m+1)\times(m+1)$, it follows that $(\Delta^{(m)})^{m+1}=0$. Thus the singular part of (4.18) can be written as the finite sum

$$\frac{P^{(m)}}{z-\Psi^{(m)}} = P^{(m)} \sum_{k=0}^{m} \frac{[\Delta^{(m)}(z)]^k}{[z-e^{-\gamma^{(m)}}]^{k+1}} \ . \tag{4.26}$$

The expression (4.16) for the time correlation with respect to the mth mode can be written then as

$$(B|\Sigma^{(m)}(t)|A) = \frac{1}{2\pi i}\oint_{z=e^{-\gamma^{(m)}}} dz\, z^t \sum_{i'\ge i=0}^{m+1} \sum_{j\ge j'=0}^{m+1} (B+U_b^\dagger R_B^{(m+1)}|\beta_i,\tilde{\beta}_j)(\tilde{\beta}_i,\beta_j|[P^{(m)}+\mathcal{C}^{(m)}(z)])$$
$$\times \sum_{k=0}^{m} \frac{[\Delta^{(m)}(z)]^k}{[z-e^{-\gamma^{(m)}}]^{k+1}} [P^{(m)}+\mathcal{D}^{(m)}(z)]|\beta_{i'},\tilde{\beta}_{j'})$$
$$\times(\tilde{\beta}_{i'},\beta_{j'}|A+U_b R_A^{(m+1)}) \ . \tag{4.27}$$

Since the pole at $z=e^{-\gamma^{(m)}}$ has been made explicit in (4.27), the integration may be evaluated by the Cauchy residue theorem.

Our construction using the Bernoulli basis (with remainder) introduces self-similar functions which enable us to obtain compact expressions for the first three exponentially decaying modes in the time correlation function of the baker transformation. The expression for the mth mode is applicable for any probability density and observable if the m-times derivative with respect to x of the probability density and the m-times derivative with respect to y of the observable belong to L_2.

In the calculation of the time correlation function, it is convenient to consider a set of projection operators which project out each exponentially decaying mode. For the baker transformation it is necessary to consider projection operators which project out an eigenspace instead of an eigenstate. The formalism of the projection operators for the eigenspaces was introduced by the Brussels group and is called subdynamics. It is discussed in Appendix D.

The explicit evaluation of (4.27) is tedious for $m>0$. Here we give the results for $m=0$, 1, and 2. Some details of the calculations are given in Appendix E.

(1) $m=0$ case:

$$(B|\Sigma^{(0)}(t)|A) = (B|1,1)(1,1|A) \ . \tag{4.28}$$

(2) $m=1$ case:

$$(B|\Sigma^{(1)}(t)|A) = e^{-\gamma^{(1)}t}\left[(B|\beta_1,1)+\tfrac{1}{2}(\frac{\partial^2 B}{\partial y^2}|1,w_{\lambda_2})\right](1,1|\frac{\partial A}{\partial x})$$
$$+(\frac{\partial B}{\partial y}|1,1)\left[(1,\beta_1|A)+\tfrac{1}{2}(w_{\lambda_2},1|\frac{\partial^2 A}{\partial x^2})\right]+\tfrac{1}{16}te^{-\gamma^{(1)}(t-1)}(\frac{\partial B}{\partial y}|1,1)(1,1|\frac{\partial A}{\partial x}) \ . \tag{4.29}$$

(3) $m=2$ case:

$$(B|\Sigma^{(2)}(t)|A) = e^{-\gamma^{(2)}t}[(\partial_y^2 B|1,1)\{(1,\beta_2|A)+\tfrac{1}{2}(w_{\lambda_2},\beta_1|\partial_x^2 A)-\tfrac{1}{8}(w_\eta,1|\partial_x^3 A)+\tfrac{1}{16}(w'_{\lambda_2},\ |\partial_x^3 A)\}$$
$$+\{(\partial_y B|\beta_1,1)+\tfrac{1}{2}(\partial_y^3 B|1,w_{\lambda_2})\}\{(1,\beta_1|\partial_x A)+\tfrac{1}{2}(w_{\lambda_2},1|\partial_x^3 A)\}$$
$$+\{(B|\beta_2,1)+\tfrac{1}{2}(\partial_y^2 B|\beta_1,w_{\lambda_2})-\tfrac{1}{8}(\partial_y^3 B|1,w_\eta)+\tfrac{1}{16}(\partial_y^3 B|1,w'_{\lambda_2})\}(1,1|\partial_x^2 A)]$$
$$+te^{-\gamma^{(2)}(t-1)}\tfrac{1}{8}[(\partial_y^2 B|1,1)\{(1,\beta_1|\partial_x A)+\tfrac{1}{2}(w_{\lambda_2},1|\partial_x^3 A)\}$$
$$+\{(\partial_y B|\beta_1,1)+\tfrac{1}{2}(\partial_y^3 B|1,w_{\lambda_2})\}(1,1|\partial_x^2 A)]$$
$$+\frac{t(t-1)}{2}e^{-\gamma^{(2)}(t-2)}\tfrac{1}{64}(\partial_y^2 B|1,1)(1,1|\partial_x^2 A) \ , \tag{4.30}$$

where

$$w_f(x) \equiv \frac{1}{1-\bar{U}_B^\dagger/2}f(x) \ , \tag{4.31}$$

$$w'_f(x) \equiv -\frac{1}{(1-\bar{U}_B^\dagger/2)^2}f(x) \ , \tag{4.32}$$

$$\lambda_n(x) \equiv -\int_0^1 dx' r_1(x')[\beta_n(x')-\beta_n(x'-x)] \ , \tag{4.33}$$

$$\eta(x) \equiv \int_0^1 dx' w_{\lambda_2}(2x')r_1(x')\beta_1(x'-x) \ . \tag{4.34}$$

The explicit form of $\lambda_2(x)$ is given as

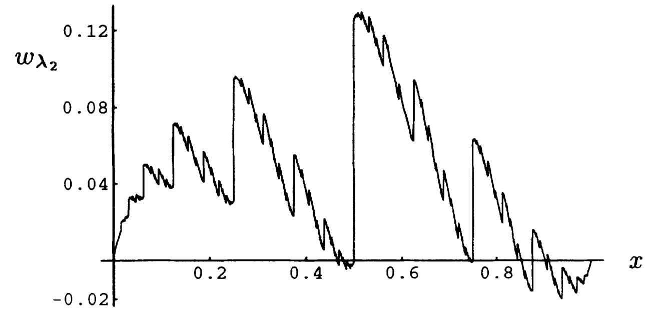

FIG. 6. The function $w_{\lambda_2}(x)$.

$$\lambda_2(x)=\begin{cases} -x^2/2+x/4 & \text{if } 0\le x<\frac{1}{2} \\ (1-x)^2/2-(1-x)/4 & \text{if } \frac{1}{2}\le x<1 \; . \end{cases} \tag{4.35}$$

The function $w_{\lambda_2}(x)$ is a self-similar function which is illustrated in Fig. 6.

V. THE MULTIBAKER TRANSFORMATION

Gaspard's multibaker map [18,22] is the two-dimensional extension of the multi-Bernoulli map to an area-preserving transformation. We again have diffusion as for the multi-Bernoulli map but here the transformation is time reversible. Diffusion is the simplest transport property but its elucidation in, for example, a Hamiltonian gas system proceeds from the Hamiltonian through a series of assumptions and approximations to the phenomenological diffusion equation governing the evolution of an inhomogeneous density. Thus, the kinetic behavior as an exact consequence of the dynamics demonstrates that extramechanical elements are not a necessary condition for an irreversible description of deterministic time-reversible systems.

The multibaker map is constructed on a chain of squares along the X axis. The first map acts on the squares

$$S_1^{(q)}=\{(X,y):\; q-\tfrac{1}{2}\le X<q+\tfrac{1}{2}, 0\le y<1\}\;, \tag{5.1}$$

as

$$\Phi_1(X,y)=\begin{cases} \left[2X-q+\frac{1}{2},\frac{y}{2}\right]\;, & q-\frac{1}{2}\le x<q,\; 0\le y<1 \\ \left[2X-q-\frac{1}{2},\frac{y}{2}+\frac{1}{2}\right]\;, & q\le x<q+\frac{1}{2},\; 0\le y<1\;. \end{cases} \tag{5.2}$$

The second map $\Phi_2(X,y)$ acts on squares shifted $\frac{1}{2}$ to the right

$$S_2^{(q)}=\{(X,y):\; q\le X<q+1, 0\le y<1\}\;, \tag{5.3}$$

as

$$\Phi_2(X,y)=\begin{cases} \left[2X-q,\frac{y}{2}\right]\;, & q\le x<q+\frac{1}{2},\; 0\le y<1 \\ \left[2X-q-1,\frac{y}{2}+\frac{1}{2}\right]\;, & q+\frac{1}{2}\le x<q+1,\; 0\le y<1\;. \end{cases} \tag{5.4}$$

The multibaker transformation is given by the composition of these two maps:

$$\Phi(X,y)=\Phi_2(X,y)\circ\Phi_1(X,y)\;, \tag{5.5}$$

which then maps regions of each cell to the adjoining cells. We obtain from the above definitions that

$$\Phi(X,y)=\begin{cases} \left[4X-3q-1,\frac{y}{4}+\frac{3}{4}\right], & q\le X\le q+\frac{1}{4},\; 0\le y\le 1 \\ \left[4X-3q-1,\frac{y}{4}+\frac{1}{4}\right], & q+\frac{1}{4}\le X\le q+\frac{1}{2},\; 0\le y\le 1 \\ \left[4X-3q-2,\frac{y}{4}+\frac{1}{2}\right], & q+\frac{1}{2}\le X\le q+\frac{3}{4},\; 0\le y\le 1 \\ \left[4X-3q-2,\frac{y}{4}\right], & q+\frac{3}{4}\le X\le q+1,\; 0\le y\le 1\;. \end{cases} \tag{5.6}$$

The map is illustrated in Fig. 7.

Note that the projection of the multibaker map onto the X axis is the multi-Bernoulli map which has been studied in Sec. III. The multibaker map has the Lyapunov exponent $\ln 4$ associated with the stretching in the X direction and $\ln\frac{1}{4}$ associated with the contraction in the y direction.

Since Φ is invertible, we may immediately write down the action of the Frobenius-Perron operator as

$$\rho(X,y;t+1)=U_{\text{mb}}\rho(X,y;t)=\rho(\Phi^{-1}(X,y);t)=\begin{cases}\rho\left[\frac{X}{4}+\frac{3q}{4}-\frac{1}{4},4y;t\right], & q\le X\le q+1,\ 0\le y\le\frac{1}{4}\\ \rho\left[\frac{X}{4}+\frac{3q}{4}+\frac{1}{4},4y-1;t\right], & q\le X\le q+1,\frac{1}{4}\le y\le\frac{1}{2}\\ \rho\left[\frac{X}{4}+\frac{3q}{4}+\frac{1}{2},4y-2;t\right], & q\le X\le q+1,\frac{1}{2}\le y\le\frac{3}{4}\\ \rho\left[\frac{X}{4}+\frac{3q}{4}+1,4y-3;t\right], & q\le X\le q+1,\frac{3}{4}\le y\le 1\ .\end{cases} \tag{5.7}$$

Decomposing X into its integer q and fractional part x, we separate the motion among cells from the internal motion in the cells through the discrete Fourier transform with respect to q as we did for the multi-Bernoulli map using (3.3). The modes ρ_s evolve under the map then as

$$\rho_s(x,y;t+1)=\begin{cases}e^{-i(2\pi/L)s}\rho_s\left[\frac{x}{4}+\frac{3}{4},4y;t\right], & 0\le y<\frac{1}{4}\\ \rho_s\left[\frac{x}{4}+\frac{1}{4},4y-1;t\right], & \frac{1}{4}\le y<\frac{1}{2}\\ \rho_s\left[\frac{x}{4}+\frac{1}{2},4y-2;t\right), & \frac{1}{2}\le y<\frac{3}{4}\\ e^{i(2\pi/L)s}\rho_s\left[\frac{x}{4},4y-3;t\right], & \frac{3}{4}\le y<1\ .\end{cases} \tag{5.8}$$

This transformation corresponds to the square of the transformation

$$U_s\rho_s(x,y;t)=\begin{cases}e^{-i(\pi/L)s}\rho_s\left[\frac{x}{2}+\frac{1}{2},2y;t\right], & 0\le y<\frac{1}{2}\\ e^{i(\pi/L)s}\rho_s\left[\frac{x}{2},2y-1;t\right], & \frac{1}{2}\le y<1\ .\end{cases} \tag{5.9}$$

We may rewrite this as

$$U_s\rho_s(x,y;t)=\cos\left[\frac{\pi s}{L}\right]U_x\left\{1+r_1(x)r_1(y)+i\tan\left[\frac{\pi s}{L}\right][r_1(x)+r_1(y)]\right\}\times U_y^{\dagger}\rho_s(x,y;t)\ , \tag{5.10}$$

which for $s=0$ is the baker transformation.

As for the baker transformation, the operator U_s satisfies the following pair of intertwining relations analogous to (4.3) and (4.4):

$$\frac{\partial^M}{\partial x^M}U_s^n A_s(x,y)=\left[\frac{1}{2^n}\right]^M U_s^n\frac{\partial^M}{\partial x^M}A_s(x,y)\ , \tag{5.11}$$

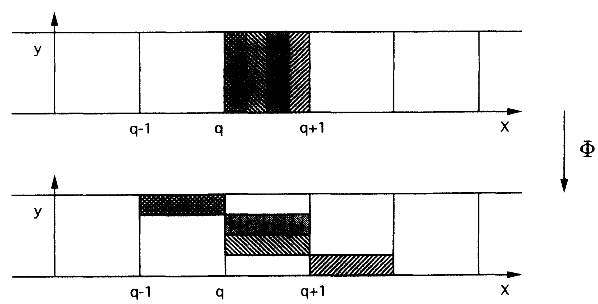

FIG. 7. The multibaker transformation.

if $A_s(x,y)$ is at least M-times differentiable with respect to x and

$$\frac{\partial^M}{\partial y^M}(U_s^\dagger)^n B_s(x,y)=\left[\frac{1}{2^n}\right]^M (U_s^\dagger)^n \frac{\partial^M}{\partial y^M}B_s(x,y) , \tag{5.12}$$

if $B_s(x,y)$ is at least M-times differentiable with respect to y.

The two-dimensional Bernoulli basis used for the baker transformation in Sec. IV is also used here. The decomposition (5.10) splits the Frobenius-Perron operator as

$$U_s=U_{s0}+\delta U_s , \tag{5.13}$$

where

$$U_{s0}=e^{-\gamma_s^{(0)}}\overline{U}_x\overline{U}_y^\dagger , \tag{5.14a}$$

$$\delta U_s=\overline{U}_x\{r_1(x)r_1(y)+i\tan(\pi s/L)[r_1(x)+r_1(y)]\}\overline{U}_y^\dagger . \tag{5.14b}$$

The operator U_{s0} is diagonal with respect to a two-dimensional Bernoulli basis and δU_s is off-diagonal. The matrix elements are given in Appendix F. Again, we have triangularity for the matrix elements but here the off-diagonal part allows for more transitions than in the case of the baker map.

A. Time evolution of correlation functions

We consider the time evolution of the correlation function of the observables $B(q+x,y)$ and $A(q+x,y)$, i.e., $(B|U_{\rm mb}^t|A)_F$, where the subscript F denotes the inner product in the full configuration space. Using the resolvent formalism, we have

$$\begin{aligned}(B|U_{\rm mb}^t|A)_F&=\frac{1}{L}\sum_s (B_s|U_s^t|A_s)\\ &=\frac{1}{L}\sum_s\frac{1}{2\pi i}\oint_{|z|=1+\epsilon}dz\, z^t(B_s|\frac{1}{z-U_s}|A_s) .\end{aligned} \tag{5.15}$$

For a smooth probability density with respect to the dilating direction and a smooth observable with respect to the contracting direction, i.e., $\partial_x^M A_s,\partial_y^M B_s\in L_2$, we can rewrite the resolvent $(B_s|1/(z-U_s)|A_s)$ using the Euler-Maclaurin expansions in (4.5):

$$\begin{aligned}(B_s|\frac{1}{z-U_s}|A_s)=&(R_{s,B_s}^{(M)}|z-U_s|R_{s,A_s}^{(M)})+\sum_{i=0}^{M}(R_{s,B_s}^{(M)}|\beta_i,\tilde\beta_M e)(\tilde\beta_i,\mathcal{B}_M|A_s+U_sR_{s,A_s}^{(M)})\\ &+\sum_{j=0}^{M}(B_s+U_s^\dagger R_{s,B_s}^{(M)}|\mathcal{B}_M,\tilde\beta_j)(e\tilde\beta_M,\beta_j|R_{s,A_s}^{(M)})\\ &+\sum_{i'\ge i=0}^{M}\sum_{j\ge j'=0}^{M}(B_s+U_s^\dagger R_{s,B_s}^{(M)}|\beta_i,\tilde\beta_j)(\tilde\beta_i,\beta_j|\frac{1}{z-U_s}|\beta_{i'},\tilde\beta_{j'})(\tilde\beta_{i'},\beta_{j'}|A_s+U_sR_{s,A_s}^{(M)}) ,\end{aligned} \tag{5.16}$$

where $R_{s,A_s}^{(M)}(x,y)$ and $R_{s,B_s}^{(M)}(x,y)$ are defined by replacing U_b, A, and B by U_s, A_s, and B_s, respectively, in (4.10).

From the intertwining relations (5.11) and (5.12) and the expansion analogous to (4.12), the singularities of the resolvent (the spectrum) are determined for $|z|>(\frac{1}{2})^M$ and are poles at $z=e^{-\gamma_s^{(m)}}\equiv\cos(\pi s/L)/2^m$ for $m=0,1,\ldots,M-M_s+1$, where M_s is defined in (3.22). It is the same result as for the multi-Bernoulli map except for the $m+1$ degeneracy of the mth pole.

The time correlation function is given as the sum of contributions from each of the poles and the background integral at $|z|=(\frac{1}{2})^M+\epsilon$ as

$$(B|U_{\rm mb}^t|A)_F=\frac{1}{L}\sum_s\sum_{m=0}^{M-M_s}(B_s|\Sigma_s^{(m)}(t)|A_s)+\frac{1}{L}\sum_s(B_s|\mathcal{R}_s^{(M)}|A_s) , \tag{5.17}$$

where

$$(B_s|\Sigma_s^{(m)}(t)|A_s)=\frac{1}{2\pi i}\oint_{z=e^{-\gamma_s^{(m)}}}dz\, z^t(B_s|\frac{1}{z-U_s}|A_s) \tag{5.18}$$

and

$$(B_s|\mathcal{R}_s^{(M)}|A_s)=\frac{1}{2\pi i}\oint_{|z|=1/2^M+\epsilon}dz\, z^t(B_s|\frac{1}{z-U_s}|A_s) . \tag{5.19}$$

B. Projective decomposition of the resolvent

In order to evaluate the mth decaying mode $(B_s|\Sigma_s^{(m)}(t)|A_s)$, we choose $M=m+M_s$. By substituting (5.16) into (5.18), we obtain

$$(B_s|\Sigma_s^{(m)}(t)|A_s)=\frac{1}{2\pi i}\oint_{z=e^{-\gamma_s^{(m)}}} dz\, z^t \sum_{i'\geq i=0}^{m+M_s}\sum_{j\geq j'=0}^{m+M_s}(B_s+U_s^\dagger R_{s,B_s}^{(m+M_s)}|\beta_i,\tilde{\beta}_j)$$
$$\times(\tilde{\beta}_i,\beta_j|\frac{1}{z-U_s}|\beta_{i'},\tilde{\beta}_{j'})(\tilde{\beta}_{i'},\beta_{j'}|A_s+U_sR_{s,A_s}^{(m+M_s)}) . \qquad (5.20)$$

As for the baker transformation, using the projective decomposition analogous to (4.18) of the resolvent in conjunction with the "perturbation" expansion to isolate the poles, we obtain

$$(B_s|\Sigma_s^{(m)}(t)|A_s)=\frac{1}{2\pi i}\oint_{z=e^{-\gamma_s^{(m)}}} dz\, z^t \sum_{i'\geq i=0}^{m+M_s}\sum_{j\geq j'=0}^{m+M_s}(B_s+U_s^\dagger R_{s,B_s}^{(m+M_s)}|\beta_i,\tilde{\beta}_j)$$
$$\times(\tilde{\beta}_i,\beta_j|[P_s^{(m)}+\mathcal{C}_s^{(m)}(z)]\sum_{k=0}^{m}\frac{[\Delta_s^{(m)}(z)]^k}{[z-e^{-\gamma_s^{(m)}}]^{k+1}}$$
$$\times[P_s^{(m)}+\mathcal{D}_s^{(m)}(z)]|\beta_{i'},\tilde{\beta}_{j'})$$
$$\times(\tilde{\beta}_{i'},\beta_{j'}|A_s+U_sR_{s,A_s}^{(m+M_s)}) , \qquad (5.21)$$

where $P_s^{(m)}$, $\mathcal{C}_s^{(m)}$, $\mathcal{D}_s^{(m)}$, $\Delta_s^{(m)}$, and other associated operators are defined in the transformed space by replacing U_b by U_s in the equations which define $P^{(m)}\mathcal{C}^{(m)}, \ldots$ in Sec. IV.

Since the pole at $z=e^{-\gamma_s^{(m)}}$ has been made explicit in (5.21), the integration may be evaluated by the Cauchy residue theorem. The explicit evaluation of (5.21) is tedious for $m>0$. Here we give the results for $m=0$ and 1 and $M_s=1$:

(1) $m=0$ case:

$$(B_s|\Sigma_s^{(0)}(t)|A_s)=e^{-\gamma_s^{(0)}t}(B_s|1,\tilde{\gamma}_{-s}^{(0)})(\tilde{\gamma}_s^{(0)},1|A_s) , \qquad (5.22)$$

where $\tilde{\gamma}_s^{(0)}$ is the $m=0$ left eigenstate of $\overline{U}_s$ whose explicit form is given in (3.52). The relation of the eigenmodes of the multibaker map with the eigenstates of the multi-Bernoulli map was pointed out in our previous paper [20].

(2) $m=1$ case:

$$(B_s|\Sigma_s^{(1)}(t)|A_s)=e^{-\gamma_s^{(1)}t}\Bigg[(B_s|1,\tilde{\gamma}_s^{(1)})\Bigg\{(1,\beta_1|A_s)+\frac{1}{2}i\tan\left[\frac{\pi s}{L}\right][(1,1|A_s)-(1,\beta_1|\partial_x A_s)]$$
$$+\frac{1}{2}\left\{1+2\tan^2\left[\frac{\pi s}{L}\right]\right\}(w_{s,\lambda_2},1|\partial_x^2 A_s)-\frac{1}{2}i\tan\left[\frac{\pi s}{L}\right](w_{s,\lambda_1},1|\partial_x A_s)$$
$$+\frac{1}{4}i\tan\left[\frac{\pi s}{L}\right][(-\tfrac{1}{4}w_{s,1}+w_{s,\lambda_2},\beta_1|\partial_x A_s)-\tfrac{1}{4}(-\tfrac{1}{4}w_{s,\eta_{s,1}}+w_{s,\eta_{s,\lambda_1}},1|\partial_x A_s)]$$
$$+\frac{1}{4}\tan^2\left[\frac{\pi s}{L}\right]\Bigg[(w_{s,r_1},\beta_1|\partial_x A_s)-\frac{1}{24}i\tan\left[\frac{\pi s}{L}\right](w_{s,1},1|\partial_x A_s)$$
$$+\frac{1}{4}(w_{s,\eta_{s,r_1}},1|\partial_x A_s)\Bigg]$$
$$-\frac{1}{8}i\tan\left[\frac{\pi s}{L}\right]\left\{1+\tan^2\left[\frac{\pi s}{L}\right]\right\}(w'_{-s,\lambda_1},1|\partial_x A_s)\Bigg\}$$
$$+\Bigg\{(B_s|\beta_1,1)+\frac{1}{2}i\tan\left[\frac{\pi s}{L}\right][(B_s|1,1)-(\partial_y B_s|\beta_1,1)]$$
$$+\frac{1}{2}\left\{1+2\tan^2\left[\frac{\pi s}{L}\right]\right\}(\partial_y^2 B_s|1,w_{-s,\lambda_2})-\frac{1}{2}i\tan\left[\frac{\pi s}{L}\right](\partial_y B_s|1,w_{-s,\lambda_1})$$
$$+\frac{1}{4}i\tan\left[\frac{\pi s}{L}\right][(\partial_y B_s|\beta_1,-\tfrac{1}{4}w_{-s,1}+w_{-s,\lambda_2})-\tfrac{1}{4}(\partial_y B_s|1,-\tfrac{1}{4}w_{-s,\eta_{-s,1}}+w_{-s,\eta_{-s,\lambda_1}})]$$
$$+\frac{1}{4}\tan^2\left[\frac{\pi s}{L}\right]\Bigg[(\partial_y B_s|\beta_1,w_{-s,r_1})-\frac{1}{24}i\tan\left[\frac{\pi s}{L}\right](\partial_y B_s|1,w_{-s,1})$$

$$
\left. + \frac{1}{4}(\partial_y B_s | 1, w_{-s,\eta_{-s,r_1}}) \right]
$$

$$
- \frac{1}{8} i \tan\left[\frac{\pi s}{L}\right] \left\{ 1 + \tan^2\left[\frac{\pi s}{L}\right] \right\} (\partial_y B_s | 1, w'_{-s,\lambda_1}) \Bigg\} (\tilde{\gamma}_s^{(1)}, 1 | A_s)
$$

$$
+ \frac{1}{2} \tan^2 \left[\frac{\pi s}{L}\right] (B_s | 1, \tilde{\gamma}_{-s}^{(1)})(\tilde{\gamma}_s^{(1)}, 1 | A_s) \Bigg]
$$

$$
+ t e^{-\gamma_s^{(1)} t} \frac{1}{8} \left[1 + \tan^2 \left[\frac{\pi s}{L}\right] \right] (B_s | 1, \tilde{\gamma}_{-s}^{(1)})(\tilde{\gamma}_s^{(1)}, 1 | A_s) , \tag{5.23}
$$

where the self-similar (fractal) functions $w_{s,f}(x)$, $w'_{s,f}(x)$, and $\eta_{s,f}(x)$ are defined as

$$
w_{s,f}(x) \equiv \frac{1}{1 - \overline{U}_s^\dagger e^{\gamma_s^{(0)}}/2} f(x) , \tag{5.24a}
$$

$$
w'_{s,f}(x) \equiv - \frac{1}{(1 - \overline{U}_s^\dagger e^{\gamma_s^{(0)}}/2)^2} f(x) , \tag{5.24b}
$$

$$
\eta_{s,f}(x) \equiv \int_0^1 dx' \beta_1(x'-x) r_1(x') \overline{U}_s^\dagger w_{s,f}(x') . \tag{5.24c}
$$

The subdynamics formalism for the multibaker map is discussed in Appendix G.

VI. CONCLUSIONS

For chaotic systems characterized by trajectory instability (positive Lyapunov exponent), the classical concept of a deterministic trajectory loses operational meaning but the description in terms of an ensemble characterized by a smooth probability density is solvable and manifestly displays the intrinsically irreversible nature of such systems. In this paper we showed for the baker transformation and the multibaker transformation that decay rates of the time correlation function are uniquely determined by restricting the observable and the probability density and we constructed the decaying eigenspace explicitly. After a time scale characterized by the Lyapunov time, only modes associated with diffusion dominate in the multi-Bernoulli and the multibaker map. Then the kinetic equation (diffusion equation) is valid.

In our construction of the generalized spectral representation, we found that the eigenstates (eigenspaces) have a self-similar nature. The eigenstates (eigenspaces) were written as products of self-similar functions and derivative operators. Using the self-similar functions, we obtained closed expressions for the first few exponentially decaying modes in the time correlation function of the baker and multibaker transformations.

The eigenstates of the diffusive systems we considered are of a fractal (noninteger dimension) nature. The fractality can be seen as a consequence of the eigenstate equation which can be considered as a scaling relation due to the stretching dynamics of the map. (Recently, Tasaki, Antoniou, and Suchanecki [40] have considered the fractal nature of the left eigenstates of the multi-Bernoulli map and have shown how they can be expressed in terms of solutions to DeRham's functional equation.) The fractality of the eigenstates is not just a property of the systems we considered but a general one of highly chaotic periodic systems. For example, for the standard map, we can consider motion inside a cell as a scale transformation in the highly chaotic region for large stochastic parameter. Therefore, the eigenvalue equation in the Fourier transformed space can be considered as a scale relation so that we expect similar fractality for the eigenstates of the standard map.

In the usual spectral theory in Hilbert space, highly chaotic conservative systems such as the baker and the multibaker transformation have absolutely continuous spectra (Lebesgue spectrum) [41] containing no special time scales associated with irreversibility such as decay rates or diffusion coefficients. This is reasonable, since these systems are symmetric under a time reversal transformation. By imposing the condition of differentiability on the observable and probability density with respect to the coordinate we consider, we obtain the decay rates which characterize irreversibility. This is a kind of symmetry breaking.

In this sense it is interesting to compare the chaotic system with a spin system since a spin system is a typical model of symmetry breaking. Although the Hamiltonian of the spin system is symmetric under a rotational transformation, a special direction of magnetization appears in the ordered phase. It is possible to consider a spin distribution for which another direction of magnetization appears. But because there is an infinite energy barrier, as in the Ising model, it is impossible to transform from one distribution to the other so that symmetry breaking is realized.

On the analogy of the spin system it may be possible to consider differentiability with respect to a new coordinate determining different decay rates. Clearly, the physical decaying eigenstate in the original coordinate will not be smooth in the new coordinate. The transformation from the original coordinate to the new coordinate changes the

Lebesgue measure to a singular measure [36,42]. In this sense it is physically impossible to change decay rates. This means that we already select one set of physical decay rates when we consider the original coordinate with Lebesgue measure. This is related to the reason Kolmogorov introduced his entropy to distinguish two spectrally equivalent systems [43].

Although the phase space of the maps we considered is uniformly highly chaotic, in Hamiltonian systems chaotic regions and regular regions may coexist. Since the stretching factor approaches unity near the boundaries of these regions, intermittency or anomalous diffusion [44,45] is dominant over normal diffusion for long times [46]. Since our method is based on the piecewise linearity of maps, we need to find a good piecewise-linear approximation for these systems. Intermittency and anomalous diffusion using a piecewise-linear approximation of the systems will be discussed in forthcoming papers [47].

ACKNOWLEDGMENTS

We are grateful to I. Prigogine for his suggestions, encouragement, and support. We also thank P. Gaspard, W. Saphir, S. Tasaki, and M. Yamaguti for their comments and suggestions. We acknowledge the U.S. Department of Energy, Grant. No. DE-FG05-88ER13897; the Robert Welch foundation, Grant No. F-0365, and the European Community, Contract No. PSS*0661, for support of this work.

APPENDIX A: MATRIX ELEMENTS OF $\overline{U}_s$

The matrix elements of $\overline{U}_B$ are easily calculated as

$$\langle \tilde{\beta}_j | \overline{U}_B | \beta_{j'} \rangle = \frac{1}{2^{j'}} \langle \tilde{\beta}_j | \beta_{j'} \rangle = \frac{1}{2^j} \delta_{j,j'} . \tag{A1}$$

For the matrix elements of $\overline{U}_B r_1(x)$, we use the intertwining relation between $\overline{U}_B r_1$ and the derivative operator to obtain

$$\begin{aligned}\langle \tilde{\beta}_j | \overline{U}_B r_1 | \beta_{j'} \rangle &= \langle 1 | \frac{d^j}{dx^j} \overline{U}_B r_1 | \beta_{j'} \rangle \\ &= \frac{1}{2^j} \langle 1 | \overline{U}_B r_1 \frac{d^j}{dx^j} | \beta_{j'} \rangle \\ &= \frac{1}{2^j} \langle 1 | r_1 | \beta_{j'-j} \rangle = \frac{c_{j'-j}}{2^j} ,\end{aligned} \tag{A2}$$

where

$$c_j = \langle r_1 | \beta_j \rangle = 2\beta_{j+1}(\tfrac{1}{2}) - \beta_{j+1}(1) - \beta_{j+1}(0) . \tag{A3}$$

From (3.10), using (A1) and (A2), we obtain the matrix elements of $\overline{U}_s$ as

$$\langle \tilde{\beta}_j | \overline{U}_s | \beta_{j'} \rangle = \begin{cases} e^{-\gamma_s^{(j)}} i \tan\left[\dfrac{\pi s}{L}\right] c_{j-j'} & \text{if } j' > j \\ e^{-\gamma_s^{(j)}} & \text{if } j' = j \\ 0 & \text{if } j' < j . \end{cases} \tag{A4}$$

We also need the matrix elements of $\overline{U}_s$ with respect to the remainder part in the Euler-Maclaurin expansion. They are $(j,j' \le M)$

$$\langle e_k \tilde{\beta}_M | \overline{U}_s | \beta_{j'} \rangle = 0 , \tag{A5}$$

$$\langle \tilde{\beta}_j | \overline{U}_s | \beta_{M,k} \rangle = e^{-\gamma_s^{(j)}} i \tan\left[\frac{\pi s}{L}\right] c_{M-j,k} , \tag{A6}$$

and

$$\langle e_k \tilde{\beta}_M | \overline{U}_s | \beta_{M,k'} \rangle = e^{-\gamma_s^{(M)}} \left[-\delta_{2k,k'} + i \tan\left[\frac{\pi s}{L}\right] c_{0,k'-2k} \right] , \tag{A7}$$

where

$$\begin{aligned} c_{j,k} &= \langle r_1 | \beta_{j,k} \rangle \\ &= 2\beta_{j+1,k}(\tfrac{1}{2}) - \beta_{j+1,k}(1) - \beta_{j+1,k}(0) . \end{aligned} \tag{A8}$$

According to (A4) [and (A5)] $\overline{U}_s$ is upper triangular in the Bernoulli basis.

APPENDIX B: MATRIX ELEMENTS OF U_b

The matrix elements of the diagonal part are

$$\begin{aligned}(a,b | U_{b0} | c,d) &= \langle a | \overline{U}_x | c \rangle \langle b | \overline{U}_y^\dagger | d \rangle \\ &= \langle a | \overline{U}_x | c \rangle \langle d | \overline{U}_y | b \rangle^* ,\end{aligned} \tag{B1}$$

where $a(x) = \tilde{\beta}_i(x)$ or $e_{k_x}(x)\tilde{\beta}_M(x)$, $b(y) = \beta_i(y)$ or $\mathcal{B}_{M,k_y}(y)$, $c(x) = \beta_i(x)$ or $\mathcal{B}_{M,k_x}(x)$, and $d(y) = \tilde{\beta}_i(y)$ or $e_{k_y}(y)\tilde{\beta}_M(y)$ $(i = 0,1,\ldots,M)$. From (A1), for example,

$$(\tilde{\beta}_i, \beta_j | U_{b0} | \beta_{i'}, \tilde{\beta}_{j'}) = \delta_{i,i'} \delta_{j,j'} e^{-\gamma^{(i+j)}} . \tag{B2}$$

Similarly, for the off-diagonal part,

$$\begin{aligned}(a,b | \delta U_b | c,d) &= \langle a | \overline{U}_x r_1 | c \rangle \langle b | r_1 \overline{U}_y^\dagger | d \rangle \\ &= \langle a | \overline{U}_x r_1 | c \rangle \langle d | \overline{U}_y r_1 | b \rangle^* .\end{aligned} \tag{B3}$$

From (A2), for example,

$$\begin{aligned}(\tilde{\beta}_i, \beta_j | \delta U_b | \beta_{i'}, \tilde{\beta}_{j'}) &= \langle \tilde{\beta}_i | \overline{U}_x r_1(x) | \beta_{i'} \rangle \langle \beta_j | r_1(y) \overline{U}_y^\dagger | \tilde{\beta}_{j'} \rangle \\ &= \begin{cases} e^{\gamma^{(i+j')}} c_{i'-i} c_{j-j'} & \text{if } i' \ge i,\ j \ge j' \\ 0 & \text{otherwise} . \end{cases}\end{aligned} \tag{B4}$$

Similarly, we can calculate the other matrix elements.

APPENDIX C: DERIVATION OF (4.10)

We assume that

$$\partial_x^M A(x,y), \partial_y^M B(x,y) \in L_2 .$$

Since U_b is upper triangular with respect to the Bernoulli basis of x, on the analogy of (3.18) the resolvent operator of U_b satisfies

$$\frac{1}{z-U_b}A(x,y)=\sum_{j=0}^{M}\frac{1}{z-U_b}\beta_j(x)\left\{\langle\tilde{\beta}_j|A(y)\rangle_x-\langle\tilde{\beta}_j|U_b|\mathcal{B}_M\rangle_x\langle e\tilde{\beta}_M|\frac{1}{z-U_b}|A(y)\rangle_x\right\}$$
$$-\mathcal{B}_M(x)\langle e\tilde{\beta}_M|\frac{1}{z-U_b}|A(y)\rangle_x\ , \tag{C1}$$

where

$$\langle f|A(y)\rangle_x=\int_0^1 dx\, f^*(x)A(x,y)\ . \tag{C2}$$

Similarly,

$$\int_0^1 dx\,dy\,B^*(x,y)\frac{1}{z-U_b}=\int_0^1 dx\,dy\left[\sum_{j=0}^{M}\left\{\langle B(x)|\tilde{\beta}_j\rangle_y-\langle B(x)|\frac{1}{z-U_b}|\tilde{\beta}_M e\rangle_y\langle\mathcal{B}_M|U_b|\tilde{\beta}_j\rangle_y\right\}\beta_j(y)\frac{1}{z-U_b}\right.$$
$$\left.-\langle B(x)|\frac{1}{z-U_b}|\tilde{\beta}_M e\rangle_y\mathcal{B}_M(y)\right]\ . \tag{C3}$$

Then from (C1) and (C3), we can rewrite the correlation with respect to the resolvent as

$$(B|\frac{1}{z-U_b}|A)=\int_0^1 dx\,dy\left[\sum_{i',j=0}^{M}\left\{\langle B(x)|\tilde{\beta}_j\rangle_y-\langle B(x)|\frac{1}{z-U_b}|\tilde{\beta}_M e\rangle_y\langle\mathcal{B}_M|U_b|\tilde{\beta}_j\rangle_y\right\}\right.$$
$$\times\beta_j(y)\frac{1}{z-U_b}\beta_{i'}(x)\left\{\langle\tilde{\beta}_{i'}|A(y)\rangle_x-\langle\tilde{\beta}_{i'}|U_b|\mathcal{B}_M\rangle_x\langle e\tilde{\beta}_M|\frac{1}{z-U_b}|A(y)\rangle_x\right\}$$
$$-\sum_{i'=0}^{M}\langle B(x)|\frac{1}{z-U_b}|\tilde{\beta}_M e\rangle_y\mathcal{B}_M(y)$$
$$\times\beta_{i'}(x)\left\{\langle\tilde{\beta}_{i'}|A(y)\rangle_x-\langle\tilde{\beta}_{i'}|U_b|\mathcal{B}_M\rangle_x\langle e\tilde{\beta}_M|\frac{1}{z-U_b}|A(y)\rangle_x\right\}$$
$$-\sum_{j=0}^{M}\left\{\langle B(x)|\tilde{\beta}_j\rangle_y-\langle B(x)|\frac{1}{z-U_b}|\tilde{\beta}_M e\rangle_y\langle\mathcal{B}_M|U_b|\tilde{\beta}_j\rangle_y\right\}\beta_j(y)$$
$$\times\mathcal{B}_M(x)\langle e\tilde{\beta}_M|\frac{1}{z-U_b}|A(y)\rangle_x$$
$$\left.+\langle B(x)|\frac{1}{z-U_b}|e\tilde{\beta}_M\rangle_y\mathcal{B}_M(y)(z-U_b)\mathcal{B}_M(x)\langle e\tilde{\beta}_M|\frac{1}{z-U_b}|A(y)\rangle_x\right]\ . \tag{C4}$$

Using the Euler-Maclaurin expansion and the definitions of $R_A^{(M)}(x,y)$, (4.10a), and $R_B^{(M)}(x,y)$, (4.10b), we obtain (4.9).

APPENDIX D: SUBDYNAMICS FORMALISM FOR THE BAKER TRANSFORMATION

Because of the $m+1$ degeneracy of the mth pole, it is necessary to consider $m+1$ dimensional eigenspace instead of simple eigenstates. It is convenient to introduce a set of projection operators which project out each eigenspace and the background. The projection operators are defined as the $t\to 0$ limit of each $\Sigma^{(m)}(t)$ and the background part $\mathcal{R}^{(M)}$, respectively.

$$\Pi^{(m)}\equiv\lim_{t\to 0}\Sigma^{(m)}(t)\quad\text{for } m=0,1,\ldots,M-1\ , \tag{D1a}$$

$$\Pi^{(M)}\equiv\lim_{t\to 0}\mathcal{R}^{(M)}(t)\ . \tag{D1b}$$

These projection operators satisfy the following properties:

$$\Pi^{(m)}\Pi^{(m')}=\delta_{mm'}\Pi^{(m)}\quad\text{for } m,m'=0,1,\ldots,M-1\ , \tag{D2a}$$

$$\Pi^{(m)}U_b=U_b\Pi^{(m)}\ , \tag{D2b}$$

$$\sum_{m=0}^{M}\Pi^{(m)}=I_M\ . \tag{D2c}$$

As has been shown by the Brussels-Austin group [2,10], $\Pi^{(m)}$ is decomposed as

$$\Pi^{(m)}=[P^{(m)}+C^{(m)}]A^{(m)}[P^{(m)}+D^{(m)}]\ , \tag{D3}$$

where

$$A^{(m)} \equiv P^{(m)} \Pi^{(m)} P^{(m)}, \quad \text{(D4a)}$$

$$C^{(m)} \equiv Q^{(m)} \Pi^{(m)} P^{(m)} \overline{A}^{(m)} , \quad \text{(D4b)}$$

$$D^{(m)} \equiv \overline{A}^{(m)} P^{(m)} \Pi^{(m)} Q^{(m)} , \quad \text{(D4c)}$$

where $\overline{A}^{(m)}$ is defined by

$$P^{(m)} = A^{(m)} \overline{A}^{(m)} = \overline{A}^{(m)} A^{(m)} .$$

There exists a simple relation between $\overline{A}^{(m)}$ and $D^{(m)}$ and $C^{(m)}$:

$$\overline{A}^{(m)} = P^{(m)} + D^{(m)} C^{(m)} .$$

Using the operators $A^{(m)}$, $C^{(m)}$, and $D^{(m)}$, we can write $(B|\Sigma^{(m)}(t)|A)$ as

$$(B|\Sigma^{(m)}(t)|A) = (B|[P^{(m)}+C^{(m)}][\Theta^{(m)}]^t A^{(m)}[P^{(m)}+D^{(m)}]|A) , \quad \text{(D5)}$$

where

$$\Theta^{(m)} \equiv P^{(m)} U_b P^{(m)} + P^{(m)} U_b C^{(m)} . \quad \text{(D6)}$$

[For the derivation of (D3) and (D5), see Ref. [2]].

It is convenient to use these operators to derive the explicit form of $(B|\Sigma^{(m)}(t)|A)$ especially for $m>1$. From (D3) and (D4) the explicit forms of $A^{(m)}$, $C^{(m)}A^{(m)}$, and $A^{(m)}D^{(m)}$ are obtained as

$$A^{(m)} = P^{(m)} + \sum_{k=1}^{m} \frac{1}{k!} \frac{d^k}{dz^k} [\Delta^{(m)}(z)]^k \Big|_{z=e^{-\gamma^{(m)}}} , \quad \text{(D7a)}$$

$$\begin{aligned} C^{(m)} A^{(m)} &= \sum_{k=0}^{m} \frac{1}{k!} \frac{d^k}{dz^k} \mathcal{C}^{(m)}(z) [\Delta^{(m)}(z)]^k \Big|_{z=e^{-\gamma^{(m)}}} \\ &= \sum_{k=0}^{m} \frac{1}{k!} \frac{d^k}{dz^k} \left\{ \overline{P}^{(m+1)} \mathcal{C}^{(m)}(z) + \frac{1}{z-U_b} P_y^{(m+1)} \delta U_b [P^{(m)} + \mathcal{C}^{(m)}(z)] \right\} [\Delta^{(m)}(z)]^k \Big|_{z=e^{-\gamma^{(m)}}} , \end{aligned} \quad \text{(D7b)}$$

$$\begin{aligned} A^{(m)} D^{(m)} &= \sum_{k=0}^{m} \frac{1}{k!} \frac{d^k}{dz^k} [\Delta^{(m)}(z)]^k \mathcal{D}^{(m)}(z) \Big|_{z=e^{-\gamma^{(m)}}} \\ &= \sum_{k=0}^{m} \frac{1}{k!} \frac{d^k}{dz^k} [\Delta^{(m)}(z)]^k \left\{ \mathcal{D}^{(m)}(z) \overline{P}^{(m+1)} + [P^{(m)} + \mathcal{D}^{(m)}(z)] \delta U_b P_x^{(m+1)} \frac{1}{z-U_b} \right\} \Bigg|_{z=e^{-\gamma^{(m)}}} , \end{aligned} \quad \text{(D7c)}$$

where

$$\overline{P}^{(m)} \equiv \sum_{i,j=0}^{m} |\beta_i, \tilde{\beta}_j)(\tilde{\beta}_i, \beta_j| , \quad \text{(D8a)}$$

$$P_x^{(m)} \equiv -\sum_{j=0}^{m} |\mathcal{B}_m, \tilde{\beta}_j)(\mathbf{e}\tilde{\beta}_m, \beta_j| + |\mathcal{B}_m, \tilde{\beta}_m \mathbf{e})(\mathbf{e}\tilde{\beta}_m, \mathcal{B}_m| , \quad \text{(D8b)}$$

$$P_y^{(m)} \equiv -\sum_{j=0}^{m} |\beta_j, \tilde{\beta}_m \mathbf{e})(\tilde{\beta}_j, \mathcal{B}_m| + |\mathcal{B}_m, \tilde{\beta}_m \mathbf{e})(\mathbf{e}\tilde{\beta}_m, \mathcal{B}_m| . \quad \text{(D8c)}$$

Since $A^{(m)}=P^{(m)}$ for $m=0,1,2$, we can easily obtain the explicit form of $A^{(m)}$, $C^{(m)}$, $D^{(m)}$, and $\Theta^{(m)}$ by comparing (4.28), (4.29), (4.30), and (D7):

(1) For $m=0$:

$$A^{(0)} = P^{(0)} , \quad \text{(D9a)}$$

$$\Theta^{(0)} = P^{(0)} , \quad \text{(D9b)}$$

$$C^{(0)} = 0 , \quad \text{(D9c)}$$

$$D^{(0)} = 0 . \quad \text{(D9d)}$$

(2) For $m=1$:

$$A^{(1)} = P^{(1)}, \quad \text{(D10a)}$$

$$\Theta^{(1)} = P^{(1)} U_b P^{(1)} , \quad \text{(D10b)}$$

$$(B|C^{(1)}|\beta_0, \tilde{\beta}_1) = 0 , \quad \text{(D10c)}$$

$$(B|C^{(1)}|\beta_1, \tilde{\beta}_0) = \tfrac{1}{2}(\partial_y^2 B|1, w_{\lambda_2}) , \quad \text{(D10d)}$$

$$(\tilde{\beta}_1, \beta_0|D^{(1)}|A) = 0 , \quad \text{(D10e)}$$

$$(\tilde{\beta}_0, \beta_1|D^{(1)}|A) = \tfrac{1}{2}(w_{\lambda_2}, 1|\partial_y^2 A) . \quad \text{(D10f)}$$

(3) For $m=2$:

$$A^{(2)} = P^{(2)} , \quad \text{(D11a)}$$

$$\Theta^{(2)} = P^{(2)} U_b P^{(2)} , \quad \text{(D11b)}$$

$$(B|C^{(2)}|\beta_0, \tilde{\beta}_2) = 0 , \quad \text{(D11c)}$$

$$(B|C^{(2)}|\beta_1, \tilde{\beta}_1) = \tfrac{1}{2}(\partial_y^3 B|1, w_{\lambda_2}) , \quad \text{(D11d)}$$

$$(B|C^{(2)}|\beta_2, \tilde{\beta}_0) = \tfrac{1}{2}(\partial_y^2 B|\beta_1, w_{\lambda_2}) - \tfrac{1}{8}(\partial_y^3 B|1, w_\eta) + \tfrac{1}{16}(\partial_y^3 B|1, w'_{\lambda_2}) , \quad \text{(D11e)}$$

$$(\tilde{\beta}_2, \beta_0|D^{(2)}|A) = 0 , \quad \text{(D11f)}$$

$$(\tilde{\beta}_1,\beta_1|D^{(2)}|A)=\tfrac{1}{2}(w_{\lambda_2},1|\partial_x^3 A)\ , \tag{D11g}$$

$$(\tilde{\beta}_0,\beta_2|D^{(2)}|A)=\tfrac{1}{2}(w_{\lambda_2},\beta_1|\partial_x^2 A)-\tfrac{1}{8}(w_{\eta},1|\partial_x^3 A)+\tfrac{1}{16}(w'_{\lambda_2},1|\partial_x^3 A)\ , \tag{D11h}$$

where the operators for the $m=2$ case are explicitly calculated in Appendix E.

APPENDIX E: CORRELATION FUNCTIONS FOR THE BAKER MAP

1. $m=0$ case

For $m=0$, the evaluation of (4.27) is trivial. Due to upper triangularity, only the $P^{(0)}$ component contributes. Since the singularity of the resolvent operator at $z=e^{-\gamma^{(0)}}$ is a simple pole, the summation over k just contains the $k=0$ term in (4.27). Thus we have

$$(B|\Sigma^{(0)}(t)|A)=\frac{1}{2\pi i}\oint_{z=e^{-\gamma^{(0)}}}dz\, z^t(B|\beta_0,\tilde{\beta}_0)\frac{1}{z-e^{-\gamma^{(0)}}}(\tilde{\beta}_0,\beta_0|A)=(B|1,1)(1,1|A)\ . \tag{E1}$$

2. $m=1$ case

For $m=1$ we now have a contribution from the remainder terms and the singularity of the resolvent operator at $z=e^{-\gamma^{(1)}}$ is a double pole so that there is a term for $k=0$ and 1 in (4.27). The $k=1$ term will modify the exponential damping by a coefficient linear in t.

Since $(\tilde{\beta}_i,\beta_j|\delta U_b|\beta_{i'},\tilde{\beta}_{j'})$ is only nonzero for $i<i'$ and $j>j'$, there is no δU_b between $(B|$ and $|\beta_0,\tilde{\beta}_1)$ and there is only one or no δU_b between $(B|$ and $|\beta_1,\tilde{\beta}_0)$. Similarly, there is no δU_b between $(\tilde{\beta}_1,\beta_0|$ and $|A)$ and there is only one or no δU_b between $(\tilde{\beta}_0,\beta_1|$ and $|A)$. Using these properties, we have

$$\begin{aligned}(B|\Sigma^{(1)}(t)|A)=\frac{1}{2\pi i}\oint_{z=e^{-\gamma^{(1)}}}dz\,z^t\Bigg[&\Bigg\{(B|\beta_1,\tilde{\beta}_0)-(B|\beta_0,\tilde{\beta}_2)(\tilde{\beta}_0,\beta_2|\mathcal{C}^{(1)}(z)|\beta_1,\tilde{\beta}_0)\\&-(B|\frac{1}{z-U_{b0}}|\beta_0,\tilde{\beta}_2\mathbf{e})(\tilde{\beta}_0,\mathcal{B}_2|\delta U_b|\beta_1,\tilde{\beta}_0)\Bigg\}\frac{1}{z-e^{-\gamma^{(1)}}}(\tilde{\beta}_1,\beta_0|A)\\&+(B|\beta_0,\tilde{\beta}_1)\frac{1}{z-e^{-\gamma^{(1)}}}\Bigg\{(\tilde{\beta}_0,\beta_1|A)+(\tilde{\beta}_0,\beta_2|\mathcal{D}^{(1)}(z)|\beta_2,\tilde{\beta}_0)(\tilde{\beta}_2,\beta_0|A)\\&\qquad-(\tilde{\beta}_0,\beta_1|\delta U_b|\mathcal{B}_2,\tilde{\beta}_0)(\mathbf{e}\tilde{\beta}_2,\beta_0|\frac{1}{z-U_{b0}}|A)\Bigg\}\\&+(B|\beta_0,\tilde{\beta}_1)\frac{1}{(z-e^{-\gamma^{(1)}})^2}(\tilde{\beta}_0,\beta_1|\Delta^{(1)}(z)|\beta_1,\tilde{\beta}_0)(\tilde{\beta}_1,\beta_0|A)\Bigg]\\=e^{-\gamma^{(1)}t}\Bigg[&\Bigg\{(B|\beta_1,\tilde{\beta}_0)-(B|\frac{1}{e^{-\gamma^{(1)}}-U_{b0}}|\beta_0,\tilde{\beta}_2\mathbf{e})(\tilde{\beta}_0,\mathcal{B}_2|\delta U_b|\beta_1,\tilde{\beta}_0)\Bigg\}(\tilde{\beta}_1,\beta_0|A)\\&+(B|\beta_0,\tilde{\beta}_1)\Bigg\{(\tilde{\beta}_0,\beta_1|A)-(\tilde{\beta}_0,\beta_1|\delta U_b|\mathcal{B}_2,\tilde{\beta}_0)(\mathbf{e}\tilde{\beta}_2,\beta_0|\frac{1}{e^{-\gamma^{(1)}}-U_{b0}}|A)\Bigg\}\Bigg]\\&+te^{-\gamma^{(1)}t}(B|\beta_0,\tilde{\beta}_1)(\tilde{\beta}_0,\beta z_1|\delta U_b|\beta_1,\tilde{\beta}_0)(\tilde{\beta}_1,\beta_0|A)\ ,\end{aligned} \tag{E2}$$

where we used that $(\tilde{\beta}_0,\beta_2|\mathcal{C}^{(1)}(z)|\beta_1,\tilde{\beta}_0)=(\tilde{\beta}_0,\beta_2|\mathcal{D}^{(1)}(z)|\beta_2,\tilde{\beta}_0)=0$, since $c_2=0$.

Utilizing the intertwining relation and writing the derivative operators explicitly gives

$$\begin{aligned}(B|\Sigma^{(1)}(t)|A)=e^{-\gamma^{(1)}t}\Bigg\{&\Bigg[(B|\beta_1,1)-2(\partial_y^2 B|\frac{1}{1-\bar{U}_y^{\dagger}/2}|1,\mathbf{e})\langle\mathcal{B}_2|r_1\rangle c_1\Bigg](1,1|\partial_x A)\\&+(\partial_y B|1,1)\Bigg[(1,\beta_1|A)-2c_1\langle r_1|\mathcal{B}_2\rangle(\mathbf{e},1|\frac{1}{1-\bar{U}_x/2}|\partial_x^2 A)\Bigg]\Bigg\}\\&+te^{-\gamma^{(1)}(t-1)}(\partial_y B|1,1)c_1^2(1,1|\partial_x A)\ ,\end{aligned} \tag{E3}$$

obtaining then the expression (4.29) given in Sec. IV B.

3. $m=2$ case

For $m=2$ we will derive $(B|\Sigma^{(2)}|A)$ using the subdynamics formalism which is introduced in Appendix D. First we calculate $A^{(2)}$ using (D4a). Since the terms for $k=1$ and 2 include $\Delta^{(2)}(z)$, we will calculate $\Delta^{(2)}(z)$.

From (4.19) and (4.25),

$$\Delta^{(2)}(z)=P^{(2)}\delta U_b P^{(2)} + P^{(2)}\delta U_b Q^{(2)}\frac{1}{z-Q^{(2)}U_bQ^{(2)}}Q^{(2)}\delta U_b P^{(2)} . \quad \text{(E4)}$$

Since

$$(\tilde{\beta}_i,\beta_j|\delta U_b|\beta_{i'},\tilde{\beta}_{j'})$$

is only nonzero for $i<i'$ and $j>j'$, there is no transition between $P^{(2)}$ and $P^{(2)}$ through $Q^{(2)}$ so that the second term on the right hand side in (E4) vanishes. Since

$$\Delta^{(2)}(z)=P^{(2)}\delta U_b P^{(2)}$$

does not depend on z, the terms for $k=1$ and 2 in (D4a) also vanish so that $A^{(2)}=P^{(2)}$.

From the fact that there is no transition between $P^{(2)}$ and $P^{(2)}$ through $Q^{(2)}$, we can immediately obtain that $\Theta^{(2)}=P^{(2)}UP^{(2)}$. We will derive $C^{(2)}$ from (D4b), since $C^{(2)}A^{(2)}=C^{(2)}$ because of $A^{(2)}=P^{(2)}$.

(1) $(B|C^{(2)}|\beta_0,\tilde{\beta}_2)$. From the property of δU_b, there is no δU_b between $(B|$ and $|\beta_0,\tilde{\beta}_2)$ so that

$$(B|C^{(2)}|\beta_0,\tilde{\beta}_2)=0 . \quad \text{(E5)}$$

(2) $(B|C^{(2)}|\beta_1,\tilde{\beta}_1)$. From the property of δU_b, there is only one or no δU_b between $(B|$ and $|\beta_1,\tilde{\beta}_1)$. We have

$$(B|C^{(2)}|\beta_1,\tilde{\beta}_1)=(B|\beta_0,\tilde{\beta}_3)(\tilde{\beta}_0,\beta_3|\mathscr{C}^{(2)}(e^{-\gamma^{(2)}})|\beta_1,\tilde{\beta}_1) - (B|\frac{1}{e^{-\gamma^{(2)}}-U_{b0}}|\beta_0,\tilde{\beta}_3\mathbf{e}) \times(\tilde{\beta}_0,\mathscr{B}_3|\delta U_b|\beta_1,\tilde{\beta}_1) . \quad \text{(E6)}$$

Since $c_2=0$, the first term on the right hand side in (E6) vanishes. By utilizing the intertwining relation and writing the derivative operators explicitly, we obtain

$$(B|C^{(2)}|\beta_1,\tilde{\beta}_1)=-2c_1(\partial_y^3 B|\frac{1}{1-\overline{U}_y^\dagger/2}|1,\mathbf{e})\langle\mathscr{B}_2|r_1\rangle = \tfrac{1}{2}(\partial_y^3 B|1,w_{\lambda_2}) . \quad \text{(E7)}$$

(3) $(B|C^{(2)}|\beta_2,\tilde{\beta}_0)$. Similarly, there are only two, one, or no δU_0 between $(B|$ and $|\beta_2,\tilde{\beta}_0)$. We will consider the Euler-Maclaurin expansion up to the second order and will extend it up to the third order later for convenience. We have

$$(B|C^{(2)}|\beta_2,\tilde{\beta}_0)=-(B|\frac{1}{e^{-\gamma^{(2)}}-U_b}|\beta_1,\tilde{\beta}_2\mathbf{e})(\tilde{\beta}_1,\mathscr{B}_2|\delta U_b|\beta_2,\tilde{\beta}_0)+(B|\frac{1}{(e^{-\gamma^{(2)}}-U_{b0})^2}|\beta_0,\tilde{\beta}_2\mathbf{e})(\tilde{\beta}_0,\mathscr{B}_2|\delta U_b|\beta_1,\tilde{\beta}_1) \times(\tilde{\beta}_1,\beta_1|\Delta^{(2)}(e^{-\gamma^{(2)}})|\beta_2,\tilde{\beta}_0) . \quad \text{(E8)}$$

By utilizing the intertwining relation, writing the derivative operators explicitly, and extending Euler-Maclaurin expansion up to the third order for the second term on the right hand side in (E8), we obtain

$$(B|C^{(2)}|\beta_2,\tilde{\beta}_0)=-2c_1(\partial_y^2 B|\frac{1}{1-U_b}|\beta_1,\lambda_2)+4c_1^3(\partial_y^3 B|\frac{1}{(1-\overline{U}_y^\dagger/2)^2}|1,\lambda_2) . \quad \text{(E9)}$$

Here the first term on the right hand side in (E9) is rewritten as

$$\begin{aligned}-2c_1(\partial_y^2 B|\frac{1}{1-U_b}|\beta_1,\lambda_2)&=-2c_1(\partial_y^2 B|\left[1+\frac{1}{1-U_b}\delta U_b\right]\frac{1}{1-U_{b0}}|\beta_1,\lambda_2)\\ &=-2c_1(\partial_y^2 B|\frac{1}{1-\overline{U}_y^\dagger/2}|\beta_1,\lambda_2)-2c_1(\partial_y^2 B|\frac{1}{1-U_b}\delta U_b\frac{1}{1-\overline{U}_y^\dagger/2}|\beta_1,\lambda_2)\\ &=-2c_1(\partial_y^2 B|\beta_1,w_{\lambda_2})-2c_1^2(\partial_y^2 B|\frac{1}{1-\overline{U}_y^\dagger}|1,r_1\overline{U}_y^\dagger w_{\lambda_2}) ,\end{aligned} \quad \text{(E10)}$$

where we used $\overline{U}_x\beta_1(x)=\tfrac{1}{2}\beta_1(x)$ and $\delta U_b\beta_1(x)\lambda_2(y)=c_1r_1(y)\overline{U}_y^\dagger\lambda_2(y)$. Using the Euler-Maclaurin expansion for the second term, we obtain

$$-2c_1(\partial_y^2 B|\frac{1}{1-U_b}|\beta_1,\lambda_2)=-2c_1(\partial_y^2 B|\beta_1,w_{\lambda_2})-2c_1^2(\partial_y^3 B|w_\eta) . \quad \text{(E11)}$$

By substituting the above equation into (E9) and using the definition of $w'_{\lambda_2}(y)$, we obtain

$$(B|C^{(2)}|\beta_2,\tilde{\beta}_0)=\tfrac{1}{2}(\partial_y^2 B|\beta_1,w_{\lambda_2})-\tfrac{1}{8}(\partial_y^3 B|1,w_\eta)+\tfrac{1}{16}(\partial_y^3 B|1,w'_{\lambda_2}) . \quad \text{(E12)}$$

Similarly, we can calculate $D^{(2)}$. By substituting the explicit forms of $A^{(2)}$, $\Theta^{(2)}$, $C^{(2)}$ and $D^{(2)}$ into (D5), we obtain (4.30).

APPENDIX F: MATRIX ELEMENTS OF U_s

The matrix elements of the diagonal part are

$$(\tilde{\beta}_i,\beta_j|U_{s0}|\beta_{i'},\tilde{\beta}_{j'})=\delta_{i,i'}\delta_{j,j'}e^{-\gamma_s^{(i+j)}}, \tag{F1}$$

where

$$e^{-\gamma_s^{(i+j)}}\equiv\frac{\cos\left[\frac{\pi s}{L}\right]}{2^{i+j}}. \tag{F2}$$

For the off-diagonal part,

$$(\tilde{\beta}_i,\beta_j|\delta U_s|\beta_{i'},\tilde{\beta}_{j'})=\begin{cases}e^{-\gamma_s^{(i+j')}}\left\{c_{i'-i}c_{j-j'}+i\tan\left[\frac{\pi s}{L}\right][c_{i'-i}\delta_{j,j'}+\delta_{i,i'}c_{j-j'}]\right\} & i'\ge i,\ j\ge j'\\ 0 \quad \text{otherwise}, & \end{cases} \tag{F3}$$

where c_j is given in (A3).

APPENDIX G: SUBDYNAMICS FORMALISM FOR THE MULTI-BAKER TRANSFORMATION

Using the subdynamics formalism introduced in Appendix D, we can write $(B_s|\Sigma_s^{(m)}(t)|A_s)$ as,

$$(B_s|\Sigma_s^{(m)}(t)|A_s)=(B_s|[P_s^{(m)}+C_s^{(m)}][\Theta_s^{(m)}]^tA_s^{(m)}[P_s^{(m)}+D_s^{(m)}]|A_s), \tag{G1}$$

where operators $A_s^{(m)}$, $C_s^{(m)}$, $D_s^{(m)}$, and $\Theta_s^{(m)}$ are defined by replacing $P^{(m)}Q^{(m)}$ and U_b by $P_s^{(m)}Q_s^{(m)}$ and U_s in (D4) and (D6). The explicit forms of the operators for $m=0$ and 1 are calculated as follows.

(1) For $m=0$,

$$A_s^{(0)}=P_s^{(0)}, \tag{G2a}$$

$$\Theta_s^{(0)}=P_s^{(0)}e^{-\gamma_s^{(0)}}, \tag{G2b}$$

$$(B_s|P_s^{(0)}+C_s^{(0)}|\beta_0,\tilde{\beta}_0)=(B_s|1,\tilde{\gamma}_{-s}^{(0)}), \tag{G2c}$$

$$(\tilde{\beta}_0,\beta_0|P_s^{(0)}+D_s^{(0)}|A_s)=(\tilde{\gamma}_s^{(0)},1|A_s). \tag{G2d}$$

(2) For $m=1$,

$$(\tilde{\beta}_0,\beta_1|A_s^{(1)}|\beta_0,\tilde{\beta}_1)=1, \tag{G3a}$$

$$(\tilde{\beta}_0,\beta_1|A_s^{(1)}|\beta_1,\tilde{\beta}_0)=\frac{1}{2}\tan^2\left[\frac{\pi s}{L}\right], \tag{G3b}$$

$$(\tilde{\beta}_1,\beta_0|A_s^{(1)}|\beta_0,\tilde{\beta}_1)=0, \tag{G3c}$$

$$(\tilde{\beta}_1,\beta_0|A_s^{(1)}|\beta_1,\tilde{\beta}_0)=1, \tag{G3d}$$

$$(\tilde{\beta}_0,\beta_1|\Theta_s^{(1)}|\beta_0,\tilde{\beta}_1)=e^{-\gamma_s^{(1)}}, \tag{G3e}$$

$$(\tilde{\beta}_0,\beta_1|\Theta_s^{(1)}|\beta_1,\tilde{\beta}_0)=\frac{e^{-\gamma_s^{(1)}}}{8}\left[1+\tan^2\left[\frac{\pi s}{L}\right]\right], \tag{G3f}$$

$$(\tilde{\beta}_1,\beta_0|\Theta_s^{(1)}|\beta_0,\tilde{\beta}_1)=0, \tag{G3g}$$

$$(\tilde{\beta}_1,\beta_0|\Theta_s^{(1)}|\beta_1,\tilde{\beta}_0)=e^{-\gamma_s^{(1)}}, \tag{G3h}$$

$$(B_s|P_s^{(1)}+C_s^{(1)}|\beta_0,\beta_1)=(B_s|1,\tilde{\gamma}_s^{(1)}), \tag{G3i}$$

$$
\begin{aligned}
(B_s|C_s^{(1)}|\beta_1,\tilde{\beta}_0) = {} & \frac{1}{2} i \tan\left[\frac{\pi s}{L}\right] [(B_s|1,1) - (\partial_y B_s|\beta_1,1)] \\
& + \frac{1}{2}\left\{1 + 2\tan^2\left[\frac{\pi s}{L}\right]\right\} (\partial_y^2 B_s|1,w_{-s,\lambda_2}) - \frac{1}{2} i \tan\left[\frac{\pi s}{L}\right] (\partial_y B_s|1,w_{-s,\lambda_1}) \\
& + \frac{1}{4} i \tan\left[\frac{\pi s}{L}\right] [(\partial_y B_s|\beta_1, -\tfrac{1}{4}w_{-s,1} + w_{-s,\lambda_2}) - \tfrac{1}{4}(\partial_y B_s|1, -\tfrac{1}{4}w_{-s,\eta_{-s,1}} + w_{-s,\eta_{-s,\lambda_1}})] \\
& + \frac{1}{4}\tan^2\left[\frac{\pi s}{L}\right] \left[(\partial_y B_s|\beta_1, w_{-s,r_1}) - \frac{1}{24} i \tan\left[\frac{\pi s}{L}\right](\partial_y B_s|1,w_{-s,1}) + \tfrac{1}{4}(\partial_y B_s|1,w_{-s,\eta_{-s,r_1}})\right] \\
& - \frac{1}{8} i \tan\left[\frac{\pi s}{L}\right] \left\{1 + \tan^2\left[\frac{\pi s}{L}\right]\right\} (\partial_y B_s|1,w'_{-s,\lambda_1}) \; , \qquad \text{(G3j)}
\end{aligned}
$$

$$
(\tilde{\beta}_1,\beta_0|P_s^{(1)} + D_s^{(1)}|A_s) = (\tilde{\gamma}_s^{(1)},1|A_s) \; , \qquad \text{(G3k)}
$$

$$
\begin{aligned}
(\tilde{\beta}_0,\beta_1|D_s^{(1)}|A_s) = {} & \frac{1}{2} i \tan\left[\frac{\pi s}{L}\right] [(1,1|A_s) - (1,\beta_1|\partial_x A_s)] \\
& + \frac{1}{2}\left\{1 + 2\tan^2\left[\frac{\pi s}{L}\right]\right\} (w_{s,\lambda_2},1|\partial_x^2 A_s) - \frac{1}{2} i \tan\left[\frac{\pi s}{L}\right] (w_{s,\lambda_1},1|\partial_x A_s) \\
& + \frac{1}{4} i \tan\left[\frac{\pi s}{L}\right] [(-\tfrac{1}{4}w_{s,1} + w_{s,\lambda_2},\beta_1|\partial_x A_s) - \tfrac{1}{4}(-\tfrac{1}{4}w_{s,\eta_{s,1}} + w_{s,\eta_{s,\lambda_1}},1|\partial_x A_s)] \\
& + \frac{1}{4}\tan^2\left[\frac{\pi s}{L}\right] \left[(w_{s,r_1},\beta_1|\partial_x A_s) - \frac{1}{24} i \tan\left[\frac{\pi s}{L}\right](w_s,1|\partial_x A_s) + \tfrac{1}{4}(w_{s,\eta_{s,r_1}},1|\partial_x A_s)\right] \\
& - \frac{1}{8} i \tan\left[\frac{\pi s}{L}\right] \left\{1 + \tan^2\left[\frac{\pi s}{L}\right]\right\} (w'_{-s,\lambda_1},1|\partial_x A_s) \; . \qquad \text{(G3l)}
\end{aligned}
$$

---

[1] I. Prigogine, *From Being to Becoming* (Freeman, San Francisco, 1980).
[2] H. H. Hasegawa and W. C. Saphir, Phys. Rev. A **46**, 7401 (1992).
[3] P. Gaspard, J. Phys. A **25**, L483 (1992).
[4] I. Antoniou and S. Tasaki, J. Phys. A **26**, 73 (1992).
[5] D. Ruelle, Phys. Rev. Lett. **56**, 405 (1986).
[6] M. Pollicott, Ann. Math. **131**, 331 (1990).
[7] In this paper we will not consider the cases of intermittency or anomalous diffusion, which display power law decay.
[8] V. Baladi and G. Keller, Commun. Math. Phys. **127**, 459 (1990).
[9] H. Rugh, Nonlinearity **5**, 1237 (1992).
[10] C. George, Physica **37**, 182 (1967); I. Progogine, C. George, F. Henin, and L. Rosenfeld, Chem. Scr. **45**, 5 (1973).
[11] A. Lasota and M. Mackey, *Probabilistic Properties of Deterministic Systems* (Cambridge University Press, Cambridge, England, 1985).
[12] H. Mori, B. So, and T. Ose, Prog. Theor. Phys. **66**, 1266 (1981).
[13] M. Dörfle, J. Stat. Phys. **40**, 92 (1985).
[14] G. Roepstorff (unpublished).
[15] I. Dana, Physica D **39**, 205 (1989).
[16] F. Christiansen, G. Paladin, and H. H. Rugh, Phys. Rev. Lett. **65**, 2087 (1990).
[17] R. Artuso, Phys. Lett. A **160**, 528 (1991).
[18] P. Gaspard, J. Stat. Phys. **68**, 673 (1992).
[19] W. C. Saphir and H. H. Hasegawa, Phys. Lett. A **171**, 317 (1992).
[20] H. H. Hasegawa and D. J. Driebe, Phys. Lett. A **168**, 18 (1992); P. Gaspard, *ibid.* **168**, 13 (1992).
[21] H. H. Hasegawa and D. J. Driebe, Phys. Lett. A **176**, 193 (1993).
[22] D. J. Driebe, Ph.D. dissertation, University of Texas at Austin, 1993.
[23] D. Ruelle, *Chaotic Evolution and Strange Attractors* (Cambridge University Press, Cambridge, England, 1989).
[24] M. Hata, J. Math. Kyoto Univ. **25**, 357 (1985).
[25] E. C. G. Sudarshan, Phys. Rev. Lett. **10**, 277 (1963).
[26] M. Reed and B. Simon, *Methods of Modern Mathematical Physics* (Academic, Orlando, 1980), Vol. 1.
[27] P. Halmos, *A Hilbert Space Problem Book* (Van Nostrand, Princeton, 1967).
[28] S. Isola, Commun. Math. Phys. **116**, 343 (1988).
[29] A. Erdélyi, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953), Vol. 1.
[30] J. Mikusiński and T. Boehme, *Operational Calculus* (Pergamon, Oxford, 1987).
[31] I. Gelfand and G. Shilov, *Generalized Functions* (Academic, New York, 1968).
[32] I. Antoniou and I. Prigogine, Nuovo Cimento **219**, 93

(1992).
[33] G. H. Hardy, *Divergent Series* (Oxford Univeristy, New York, 1949).
[34] S. Grossmann and H. Fujisaka, Phys. Rev. A **26**, 1779 (1982); S. Thomae, in *Statics and Dynamics of Nonlinear Systems,* edited by G. Benedek *et al.* (Springer, Berlin, 1983).
[35] T. Takagi, Proc. Phys. Math. Soc. Jpn. **1**, 176 (1903).
[36] M. Hata and M. Yamaguti, Jpn. J. Appl. Math. **1**, 183 (1984).
[37] T. Tél, Phys. Lett. A **119**, 65 (1986).
[38] I. Antoniou and S. Tasaki, Physica A **190**, 303 (1992).
[39] C. Obcema and E. Brändas, Ann. Phys. (N.Y.) **151**, 383 (1983).
[40] S. Tasaki, I. Antoniou, and Z. Suchanecki, Phys. Lett. A **179**, 97 (1993).
[41] V. I. Arnold, *Ergodic Problems of Classical Mechanics* (translated by A. Avez) (Benjamin, New York, 1968).
[42] W. C. Saphir (private communication).
[43] P. Billingsley, *Ergodic Theory and Information* (Wiley, New York, 1965).
[44] R. Artuso, G. Casati, and R. Lombardi, Phys. Rev. Lett. **71**, 62 (1993).
[45] X.-J. Wang and C.-K. Hu, Phys. Rev. E **48**, 728 (1993), and references therein.
[46] Y. Aizawa, Y. Kikuchi, T. Harayama, K. Yamamoto, M. Ota, and K. Tanaka, Prog. Theor. Phys. Suppl. **98**, 36 (1989).
[47] H. H. Hasegawa and E. Luschei, Phys. Lett. A **186**, 193 (1994).

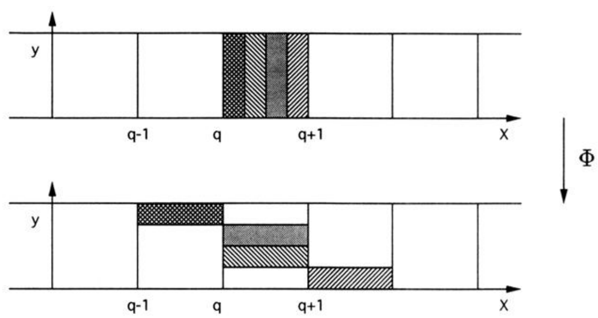

FIG. 7. The multibaker transformation.